CFT and Elliptic Cohomology

Lookout for A physical explanation of Topological Modular Forms

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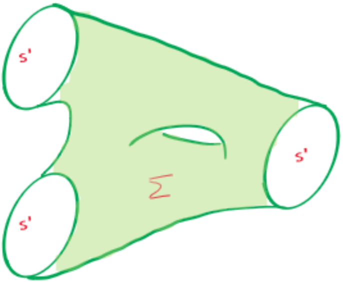
Boundary CFT

- Physicists study CFT on a specific manifold M **without** boundary (such as the plance, the torus, etc.). The method of studying these CFTs is to look at their partition functions and correlators.
- We can also study CFT on manifolds with boundary. This is called Boundary CFT [Car08].
- We associate a Hilbert spaces to the boundaries and the partition function now gives a linear map from the in-going boundary to the outgoing boundary.

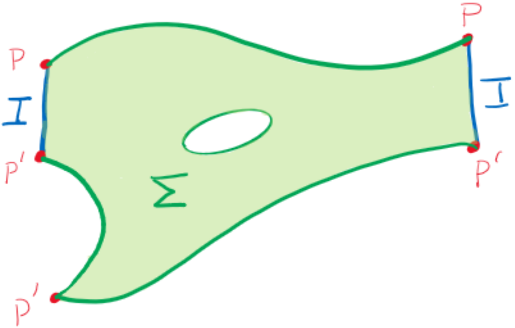


$$H(S^1) \otimes H(S^1) \xrightarrow{Z(\Sigma)} H(S^1)$$

Open-closed CFT

- Open-closed CFT is a further extension of Boundary CFT where we study manifolds with corners. Now, the boundaries themselves have boundaries. Thus these are called three-tier CFTs.
- What do CFTs assign to 0-dimensional manifolds? Von Neumann algebras! These are algebras of bounded operators of Hilbert spaces.
- To make certain constructions later work, we will need to choose parameterisations of the boundary. This is a bit unphysical. However, we will not need to if we take supersymmetric CFTs! [ST04]
- It has important applications in the study of certain critical phenomena on surfaces with boundaries in condensed matter physics. It is also a powerful tool in the study of D-branes in string theory. [Kon11]

Open-Closed CFT



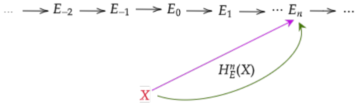
P,P' - Von Neumann Algebras V,V'

I - (V,V')-Bimodule H

Σ — linear map from H to H

Background

- [Och87]: A genus is a ring homomorphism from the cobordism ring to another ring R. An elliptic genus is a genus which takes values in the ring of modular forms.
- Every generalised cohomology theory H_E is represented by a spectrum E_n such that $H_E^n(X) = \mathrm{hTop}[X, E_n]$
- The spectrum representing the cobordism cohomology theory is called MO, and $MO^*(\mathrm{pt}) = \Omega_*$, the cobordism ring.



- Thus any map of spectra $MO \to E$, gives a genus $\Omega_* \to H_E(\mathrm{pt})$ by evaluating at a point.
- [Wit87]: Partition Functions of SCFTs give elliptic genera.

Conjecture

- [LRS95]: Defined elliptic cohomology and showed that elliptic genera come from elliptic cohomology theories (these are basically generalised cohomology theories such that $H^* (\mathrm{pt})$ is a subring of modular forms.
- [Hop95]: Defined Topological Modular Forms $\mathrm{tmf}\,(n)$ and showed that they are the universal elliptic cohomology theory.

Now, the obvious question to ask ourselves is whether TMF has a QFT interpretation. Since each SCFT leads to an elliptic genus (which itself comes from an elliptic cohomology), the universal elliptic cohomology must contain information about all SCFTs.

Conjecture [ST04]

Elements in $\mathrm{tmf}^n\,(X)$ are represented by degree n SCFTs equipped with maps into X. Also, $\mathrm{tmf}\,(n)$ is related to the space of SCFTs of degree n

Connected Components of the Space of CFTs

- Let $C(n)$ be the space of degree n SCFTs. Given any CFT Z, we can evaluate it on a torus with complex structure τ, we obtain a function $Z(\tau)$
- In [ST04], it is shown that if Z is an SCFT of degree n, then $Z(\tau)$ is an integral modular form of weight $\frac{n}{2}$. Also, on tensor producting CFTs, their partitions functions multiply, thus we have a homomorphism

$$C = \bigoplus_{n=0}^{\infty} C(n) \to \text{integral modular forms}$$

- In [Wit87], it is shown that the elliptic genus of an SCFT does not change under perturbation. Thus, it is expected that the partition function on a torus does not change within any path component in the space of SCFTs So we have a map

$$\pi_0(C) = \mathrm{hTop}[\mathrm{pt}, C] \to \text{integral modular forms}$$

- Since the algebra of integral modular forms is torsion free, the torsion elements in $\mathrm{hTop}[\mathrm{pt}, C]$ must map to 0. If we show that these are the only elements that map to 0, we get an injective homomorphism.

$$\mathrm{hTop}[\mathrm{pt}, C] \otimes \mathbb{Q} \to \text{integral modular forms}$$

- The algebra of integral modular forms is generated by 2 specific forms G_4, G_6. If we show that these arise as the partition function of some SCFTs, then the above morphism is surjective.

Some evidence for Stolz-Teichner from Modular forms

- In [Hop95], it is shown that

$$\mathrm{hTop}\left[\mathrm{pt}, \bigoplus_{n=0}^{\infty} \mathrm{tmf}\,(n)\right] \otimes \mathbb{Q} = \mathrm{tmf}^*\,(\mathrm{pt}) \otimes \mathbb{Q} = \text{integral modular forms}$$

- If the statements in the previous slide are true, then

$$\mathrm{hTop}\left[\mathrm{pt}, \bigoplus_{n=0}^{\infty} C\,(n)\right] \otimes \mathbb{Q} = \text{integral modular forms}$$

Evidence! [Bae03]

This provides some evidence that $\mathrm{tmf}\,(n) = C\,(n)$

Goals

- Understand **Witten's calculation** of how SCFTs give rise to elliptic genera [Wit87]
- Motivate from the above, the generalisation to **Stolz Teichner Conjecture** about the space of all SCFTs being the same the Topological Modular Forms (universal elliptic cohomoogy) [ST04]
- As talked about in [GJF19] and [Kit13], the space of systems (SCFTs and Gapped Systems respectively) fit together into a spectrum. The cohomology associated to this spectrum will tell us the homotopy type, and help us understand the phases of physical theories (SCFTs and Gapped Systems respectively).

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