Functorial properties of discrete Morse theory

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1 Background structures

Definition 1.1. A simplicial complex K is a pair (V_K, S_K) where V_K is a set and S_K is a collection of finite nonempty subsets of V_K such that

- $v \in V_K \implies \{v\} \in S_K$
- $\sigma \in S_K$ and $\sigma' \subseteq \sigma$ and $\sigma' \neq \emptyset \implies B \in S_K$

The elements of V_K are called the vertices of K and the elements of S_K are called the simplicies of K. For any simplex σ, define its dimension by $\dim \sigma = \operatorname{card}(\sigma) - 1$. Denote by S_K^n be subset of S_K consisting of n-dimensional simplices. A codimension k face of a simplex σ is a subset $\sigma' \subseteq \sigma$ such that $\operatorname{card}(\sigma \setminus \sigma') = k$. A finite simplicial complex is simplicial complex with a finite vertex set.

Definition 1.2. We define the category of simplicial complexes SC as

- Objects: Simplicial complexes
- A morphism $f\colon K \to L$ is a set function $f\colon V_K \to V_L$ such that for any simplex $\sigma \in S_K$, the image $f(\sigma) \in S_L$.
- Composition: We compose the underlying set functions.
- Identities: The identity morphism on K is given by id_{V_K}.

Let $\mathrm{SC_{fin}}$ denote the full subcategory consisting of the only the finite simplicial complexes.

Lemma 1.1. For $f\colon K \to L$ a simplicial map and $\sigma \in S_K$ such that $\dim \sigma > 0$, if $\dim f(\sigma) = \dim \sigma - 1$, then there are exactly two codimension 1 faces σ_1, σ_2 of σ such that $f(\sigma_1) = f(\sigma_2) = f(\sigma)$.

Proof. $f|_\sigma\colon \sigma \to f(\sigma)$ is surjective and $\operatorname{card}(f(\sigma)) = \operatorname{card}(\sigma) - 1$. Thus there will be exactly 2 elements of σ which have the same image under f, let these be v_1, v_2. Then the only two proper subsets $\sigma' \subsetneq \sigma$ which have the same image as σ are $\sigma \setminus \{v_i\}$ $(i = 1, 2)$, as removing any other point will mean that the image of the subset does not cover the image of σ. $\square$

Definition 1.3. Given a poset $(X, \leq)$ and $x_1, x_2 \in X$, we say x_2 covers x_1, denoted by $x_1 \prec x_2$ if $x_1 < x_2$ and there is no x such that $x_1 < x < x_2$.

Definition 1.4. A graded poset is a triple $(X, \leq, \dim)$ such that $(X, \leq)$ is a poset and $\dim\colon X \to \mathbb{N}$ is a monotone map such that if $x \prec x'$, then $\dim x' = \dim x + 1$.

A morphism of graded posets $f\colon (X_1, \leq_1, \dim_1) \to (X_2, \leq_2, \dim_2)$ is a montone map $f\colon (X_1, \leq_1) \to (X_2, \leq_2)$ such that for any $x \in X_1$, $\dim_2 f(x) \leq \dim_1 x$. We call the category of graded posets GrPos. Let $\mathrm{GrPos_{fin}}$ be the full subcategory of graded posets $(X, \leq, \dim)$ where X is a finite set.

There is an obvious forgetful functor $U_{\mathrm{Gr}}\colon \mathrm{GrPos} \to \mathrm{Pos}$ and let $i_{\mathrm{PC}}\colon \mathrm{Pos} \to \mathrm{Cat}$ be the inclusion. We will often leave these functor implicit.

Definition 1.5. Define a functor $S\colon \mathrm{SC} \to \mathrm{GrPos}$ which assigns a simplicial complex K to the graded poset $(S_K, \subseteq, \dim)$. This restricts to a functor $\mathrm{SC_{fin}} \to \mathrm{GrPos_{fin}}$.

Definition 1.6. Let $\mathcal{A}$ be a category and K be a simplicial complex. We define the category of $\mathcal{A}$-sheaves on K by $\mathrm{Sh}_{\mathcal{A}}(K) = [S_K, \mathcal{A}]$.

Definition 1.7. The category of $\mathcal{A}$-sheaved simplicial complex, $\mathrm{Sh}_{\mathcal{A}}\mathrm{SC}$ is defined by

- Objects: Pairs $(K, \mathcal{F})$ where K is a simplicial complex and $\mathcal{F}$ is an $\mathcal{A}$-sheaf on K.
- A morphism from $(K, \mathcal{F})$ to $(L, \mathcal{G})$ is pair (f, ϕ) where $f\colon K \to L$ is a simplicial map and $\phi\colon \mathcal{G} \circ S_f \to \mathcal{F}$ is a natural transformation.
- Given morphisms $(f, \phi) : (K, \mathcal{F}) \to (L, \mathcal{G})$ and $(g, \psi) : (L, \mathcal{G}) \to (M, \mathcal{H})$, we define the composite to be $\left(g \circ f, \phi \circ \psi_{S_f}\right)$
- The identity morphism on $(K, \mathcal{F})$ is $(\mathrm{id}_K, \mathrm{id}_{\mathcal{F}})$

$\mathrm{Sh}_{\mathcal{A}}\mathrm{SC}_{\mathrm{fin}}$ is the full subcategory of those objects whose underlying simplicial complex is finite.

There is an obvious forgetful functor $U_{\mathrm{Sh}}\colon \mathrm{Sh}_{\mathcal{A}}\mathrm{SC} \to \mathrm{SC}$. There is a similar functor for the finite case.

Definition 1.8. Given an object $V \in A$ and a simplicial complex K, we define the constant sheaf $\Delta_V : S_K \to A$ to be the functor that maps all objects to V and all morphisms to id_V.

Definition 1.9. We define the constant sheaf functor $\Delta\colon \mathrm{SC} \times \mathcal{A}^{\mathrm{op}} \to \mathrm{Sh}_{\mathcal{A}}\mathrm{SC}$ by

- For any simplicial complex K and any object $V \in \mathcal{A}$, $\Delta(K, V) = (K, \Delta_V)$
- Given a morphism $f\colon K \to L$ and a morphism $T\colon W \to V$ in $\mathcal{A}$, the morphism $(K, \Delta_V) \to (L, \Delta_W)$ defined by (f, ϕ_T) where $\phi_T\colon \Delta_W \circ S_f \to \Delta_V$ is defined by $\phi_T(\sigma) : \Delta_W(S_f(\sigma)) = W \to \Delta_V(\sigma) = V$ is just given by T.

Definition 1.10. Let R be a rig (additive inverse not necessary). An R-incidence relation on a finite simplicial complex K over a rig R is a function

$$r\colon \coprod_{n \in \mathbb{N}} S_K^n \times S_K^{n+1} \to R$$

such that for any $\sigma \in S_K^n$ and any $\sigma'' \in S_K^{n+2}$, we have

$$\sum_{\sigma \subsetneq \sigma' \subsetneq \sigma'} r(\sigma, \sigma')\, r(\sigma', \sigma'') = 0$$

This a generalisation of [Cur14, Definition 6.1.9].

Remark. Let$K$ be simplicial complex an R-incidence structure r. Let $\sigma, \sigma'' \in S_K$ be such that $\dim \sigma'' = \dim \sigma + 2$. Then $\sigma'' \setminus \sigma$ will have exactly 2 elements, let them be v_1, v_2. Thus the only σ' such that $\sigma \subsetneq \sigma' \subsetneq \sigma''$ are $\sigma'' \setminus \{v_1\}$ and $\sigma'' \setminus \{v_2\}$. Thus we can rephrase the condition in the refinition of R-incidence relation as

$$r(\sigma, \sigma'' \setminus \{v_1\})\, r(\sigma'' \setminus \{v_1\}, \sigma'') + r(\sigma, \sigma'' \setminus \{v_2\})\, r(\sigma'' \setminus \{v_2\}, \sigma'') = 0$$

Definition 1.11. The category of finite simplicial complexes with an R-incidence relation, $\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}$ is defined by

- Objects: Pairs (K, r) where K is a finite simplicial complex and r is an R-incidence relation on K
- Morphisms: A morphism $(K, r) \to (L, s)$ is a pair (f, κ) where $f\colon K \to L$ is a simplicial map and $\kappa\colon S_K \to R$ is a set function such that for any $\sigma \in S_K$ such that $\dim \sigma > 0$
  - If $\dim f(\sigma) = \dim \sigma$, then for every codimension 1 face $\sigma' \subseteq \sigma$, we have

$$\kappa(\sigma')\, r(\sigma', \sigma) = \kappa(\sigma)\, r(f(\sigma'), f(\sigma))$$

– If $\dim f(\sigma) = \dim \sigma - 1$, then let σ_1, σ_2 be the two codimension 1 faces of σ such that $f(\sigma_1) = f(\sigma_2) = f(\sigma)$, then

$$\kappa(\sigma_1) r(\sigma_1, \sigma) + \kappa(\sigma_2) r(\sigma_2, \sigma) = 0$$

- Given two morphisms $(f, \kappa) : (K, r) \to (L, s)$ and $(g, \lambda) : (L, s) \to (H, t)$, then the composition is the pair $(g \circ f, \sigma \mapsto \kappa(\sigma) \lambda(f(\sigma)))$.
- The identity of (K, r) is $(\mathrm{id}_K, \sigma \mapsto 1)$.

Let S be a multiplicative subset of R. There is a subcategory of $\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}$ such that the second component of each morphism is a set map into S, instead of all of R. We denote this by $\mathrm{IR}_R^S\mathrm{SC}_{\mathrm{fin}}$. An important special case of this is $\mathrm{IR}_R^{\{1\}}\mathrm{SC}_{\mathrm{fin}}$. Note $\mathrm{IR}_R^R\mathrm{SC}_{\mathrm{fin}} = \mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}$

Lemma 1.2 (Alternative characterisation of morphisms in $\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}$)**.** Given two objects of $\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}$, (K, r) and (L, s), a simplicial map $f : K \to L$ and a set map $\kappa : S_K \to R$ constitute a morphism $(K, r) \to (L, s)$ if and only if for any $n \in \mathbb{N}$, any $\sigma \in S_K^{n+1}$ and any $\rho \in Sn_L$, we have

$$\sum_{\sigma' \in S_K^n \mid f(\sigma') = \rho \wedge \sigma' \subseteq \sigma} \kappa(\sigma') r(\sigma', \sigma) = \begin{cases} \kappa(\sigma) s(\rho, f(\sigma)) & \text{if } f(\sigma) \in S_L^{n+1} \wedge \rho \subseteq f(\sigma) \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $\sigma \in S_K$ be such that $\dim \sigma = n + 1 > 0$

- Consider the case when $\dim f(\sigma) = \dim \sigma$. This is the same as saying $f(\sigma) \in S_L^{n+1}$, which is again equivalent to saying $f|_\sigma : \sigma \to f(\sigma)$ is a bijection. Now consider a choice of $\rho \in S_L^n$ such that $\rho \subseteq f(\sigma)$. This is equivalent to the choice of $\sigma' \subseteq \sigma$ of codimension 1, determined by $f|_\sigma^{-1}(\rho)$. Thus, the sum in the lemma reduces to a single term, and the condition becomes exactly the same as the condition for (f, κ) to be a morphism (given $\dim f(\sigma) = \dim \sigma$)
- Now consider the case when $\dim f(\sigma) = \dim \sigma - 1 = n$. Then there is a unique $\rho \in S_L^n$ such that $\rho \subseteq f(\sigma)$, given by $\rho = f(\sigma)$ (as they have the same dimension). In this case, by Lemma 1.1, there are exactly 2 codimension 1 faces of σ (let them by σ_1, σ_2) such that $f(\sigma_i) = f(\sigma) = \rho$ $(i = 1, 2)$. Thus, the sum in the lemma reduces to two terms, and the condition becomes exactly the same as the condition for (f, κ) to be a morphism (given $\dim f(\sigma) = \dim \sigma - 1$)
- In the case ρ is not contained in $f(\sigma)$ or $\dim \sigma - \dim f(\sigma) > 1$, there are no codimension 1 faces of σ such that $f(\sigma) = \rho$. Thus here, the sum in the lemma evaluates to 0, and the condition is trivially fulfilled.

□

Definition 1.12. We define the category of ordered finite simplicial complexes $\mathrm{SC}_{\mathrm{fin,or}}$ as

- Objects: Triples $(K, \leq)$ where K is a finite simplicial complex and $(V_K, \leq)$ is a total order
- A morphism $f : (K, \leq_1) \to (K, \leq_2)$ is a morphism $f : K \to L$ such that the map $V_K \to V_L$ is monotone.
- Composition is set composition
- Identities are identities as simplicial morphisms as they are automatically order preserving.

For any simplex $\sigma \in S_K$ and $0 \leq i \leq \dim \sigma$, define $\partial_i \sigma = \sigma \setminus \{v_i(\sigma)\}$ where $v_i(\sigma)$ is the i^{th} element of σ in the order induced by the inclusion $\sigma \subseteq V_K$.

Definition 1.13. If R is a ring, there is a functor $A_{\mathrm{IR,or}} : \mathrm{SC}_{\mathrm{fin,or}} \to \mathrm{IR}_R^{\{1\}}\mathrm{SC}_{\mathrm{fin}}$ defined by $(K, \leq) \mapsto (K, r)$ where r is given by (for any σ', σ such that $\dim \sigma = \dim \sigma' + 1$)

$$r(\sigma', \sigma) = \begin{cases} 0 & \text{if } \sigma' \nsubseteq \sigma \\ (-1)^i & \text{if } \sigma' = \partial_i \sigma \end{cases}$$

and a $f : (K, \leq_1) \to (L, \leq_2)$ is assigned to (f, c_1).

Proof that this is a functor. First we show that r is an incidence relation. So pick any simplices of K σ, σ'' such that $\dim \sigma'' = \dim \sigma + 2$. Then $\sigma'' \setminus \sigma$ consists of exactly 2 elements. Let these be $v_{i_1}(\sigma'')$ and $v_{i_2}(\sigma'')$ where $i_1 < i_2$. Thus the only two simplices σ' such that $\sigma \subsetneq \sigma' \subsetneq \sigma''$ are $\partial_{i_1}\sigma''$ and $\partial_{i_2}\sigma''$. Also $\partial_{i_1}\partial_{i_2}\sigma'' = \sigma$ and $\partial_{i_2-1}\partial_{i_1}\sigma'' = \sigma$

$$\begin{aligned}\sum_{\sigma \subsetneq \sigma' \subsetneq \sigma''} r(\sigma, \sigma') r(\sigma', \sigma'') &= r(\sigma, \partial_{i_1}\sigma'') r(\partial_{i_1}\sigma'', \sigma'') + r(\sigma, \partial_{i_2}\sigma'') r(\partial_{i_2}\sigma'', \sigma'') \\ &= r(\partial_{i_2-1}\partial_{i_1}\sigma'', \partial_{i_1}\sigma'') r(\partial_{i_1}\sigma'', \sigma'') + r(\partial_{i_1}\partial_{i_2}\sigma'', \partial_{i_2}\sigma'') r(\partial_{i_2}\sigma'', \sigma'') \\ &= (-1)^{i_2-1}(-1)^{i_1} + (-1)^{i_1}(-1)^{i_2} = 0\end{aligned}$$

Now we need to check if given a morphism $f\colon (K, \leq_1) \to (L, \leq_2)$, whether (f, c_1) is a morphism from $A_{\mathrm{IR,or}}(K, \leq_1) \to A_{\mathrm{IR,or}}(L, \leq_2)$. So pick a simplex $\sigma \in S_K$ such that $\dim \sigma > 0$.

- Let $\dim f(\sigma) = \dim \sigma$. Any codimension 1 face of σ is of the form $\partial_i \sigma$ for some $0 \leq i \leq \dim \sigma$. Since f is a monotone bijection between finite total orders, we have $f(v_i(\sigma)) = v_i(f(\sigma))$. Thus $\partial_i f(\sigma) = f(\partial \sigma_i)$. So we have

  $$c_1(\partial_i \sigma) r(\partial_i \sigma, \sigma) = 1 \cdot (-1)^i = c_1(f(\partial_i \sigma)) r(\partial_i f(\sigma), f(\sigma)) = c_1(f(\partial_i \sigma)) r(f(\partial_i \sigma), f(\sigma))$$

  Thus this condition is satisfies

- $\dim f(\sigma) = \dim \sigma - 1$. Let $v_{i_1}(\sigma), v_{i_2}(\sigma)$ $(i_1 < i_2)$be the two vertices of σ which have the same image. Then by monotnicity of f, these must be adjacent, i.e., $i_2 = i_1 + 1$ (for if they were not adjacent, then more than 2 elements would have had the same image and $\dim \sigma - \dim f(\sigma)$ would have been greater than 1). Then

  $$c_1(\partial_{i_1}\sigma) r(\partial_{i_1}\sigma, \sigma) + c_1(\partial_{i_1+1}\sigma) r(\partial_{i_1+1}\sigma, \sigma) = 1 \cdot (-1)^{i_1} + 1 \cdot (-1)^{i_1+1} = 0$$

  Thus this condtion is satisfied.

It is easy to see that $A_{\mathrm{IR,or}}$ preserves identites and compositions. $\square$

If R is a ring such that $1 + 1 = 0$ (i,e, R is a $\mathbb{Z}/2\mathbb{Z}$-algebra), then there is a similar functor $A_{\mathrm{IR}}\colon \mathrm{SC}_{\mathrm{fin}} \to \mathrm{IR}_R^{\{1\}}\mathrm{SC}_{\mathrm{fin}}$ which assigned the R-incidence structure defined by $r(\sigma', \sigma) = 1$ if σ' is a codimension 1 face of σ, and 0 otherwise.

Definition 1.14. We define the category $\mathrm{Sh}_{\mathcal{A}}\mathrm{IR}_R^S\mathrm{SC}_{\mathrm{fin}}$ by

- Objects: Triples $(K, \mathcal{F}, r)$ such that $(K, \mathcal{F})$ is on object of $\mathrm{Sh}_{\mathcal{A}}\mathrm{SC}_{\mathrm{fin}}$ and (K, r) is on object of $\mathrm{IR}_R^S\mathrm{SC}_{\mathrm{fin}}$.
- Morphisms are triples similar to above, and composition and identites are also done using the compositions of $\mathrm{Sh}_{\mathcal{A}}\mathrm{SC}_{\mathrm{fin}}$ and $\mathrm{IR}_R^S\mathrm{SC}_{\mathrm{fin}}$.

Lemma 1.3. We have a (1-)pullback diagram

$$\begin{tikzcd} \mathrm{Sh}_{\mathcal{A}}\mathrm{IR}_R^S\mathrm{SC}_{\mathrm{fin}} \arrow[r, "p_{\mathrm{Sh}}"] \arrow[d, "p_{\mathrm{IR}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{IR}_R^S\mathrm{SC}_{\mathrm{fin}} \arrow[d, "U_{\mathrm{IR}}"] \\ \mathrm{Sh}_{\mathcal{A}}\mathrm{SC}_{\mathrm{fin}} \arrow[r, "U_{\mathrm{Sh}}"'] & \mathrm{SC}_{\mathrm{fin}} \end{tikzcd}$$

where $p_{\mathrm{IR}}, p_{\mathrm{Sh}}$ are the obvious forgetful projection functors. This is, in fact, a $(2,1)$-pullback diagram.

2 Sheaf Cohomology

We now assume that $\mathcal{A}$ is an semiadditive category (it has finite products and coproducts and they coincide) enriched over a rig R.

Definition 2.1. For an $\mathcal{A}$-sheaved finite simplicial complex $(K, \mathcal{F})$ with a signed incidence relation r, define

$$C^n(K, \mathcal{F}, r) = \bigoplus_{\sigma \in S_K^n} \mathcal{F}(\sigma)$$

Define maps $d^n_{K,\mathcal{F},r} \colon C^n(K, \mathcal{F}, r) \to C^{n+1}(K, \mathcal{F}, r)$ by (here $\sigma \in S_K^n$ and $\sigma' \in S_K^{n+1}$)

$$\mathrm{proj}_{\mathcal{F}(\sigma')} \circ d^n_{K,\mathcal{F},r} \circ \mathrm{incl}_{\mathcal{F}(\sigma)} = \begin{cases} r(\sigma, \sigma') \mathcal{F}(\sigma \subseteq \sigma') & \text{if } \sigma \subseteq \sigma' \\ 0 & \text{otherwise} \end{cases}$$

Define $C^\bullet(K, \mathcal{F}, r) = \left(C^n(K, \mathcal{F}, r), d^n_{K,\mathcal{F},r}\right)$. This is a cochain complex concentrate in nonnegative degrees.

Proof $C^\bullet(K, \mathcal{F}, r)$ is a cochain complex. We need to prove that $d^{n+1}_{K,\mathcal{F},r} \circ d^n_{K,\mathcal{F},r} = 0$. Pick a $\sigma \in S_K^n$ and $\sigma'' \in S_K^{n+2}$. Then

$$\begin{aligned} \mathrm{proj}_{\mathcal{F}(\sigma'')} \circ d^{n+1}_{K,\mathcal{F},r} \circ d^n_{K,\mathcal{F},r} \circ \mathrm{incl}_{\mathcal{F}(\sigma)} &= \sum_{\sigma \subsetneq \sigma' \subsetneq \sigma''} r(\sigma, \sigma') r(\sigma', \sigma'') \mathcal{F}(\sigma' \subseteq \sigma'') \circ \mathcal{F}(\sigma \subseteq \sigma') \\ &= \left(\sum_{\sigma \subsetneq \sigma' \subsetneq \sigma''} r(\sigma, \sigma') r(\sigma', \sigma'') \right) \mathcal{F}(\sigma \subseteq \sigma'') = 0 \end{aligned}$$

Since this is true for any σ, σ'' and $d^{n+1}_{K,\mathcal{F},r} \circ d^n_{K,\mathcal{F},r}$ is a map from a coproduct to a product, the required statement is proved. □

Definition 2.2. Given a morphism $(f, \phi, \kappa) : (K, \mathcal{F}, r) \to (L, \mathcal{G}, s)$, we define a map $C^n(f, \phi, \kappa) : C^n(L, \mathcal{G}, s) \to C^n(K, \mathcal{F}, r)$ by (here $\rho \in S_L^n$ and $\sigma \in S_K^n$)

$$\mathrm{proj}_{\mathcal{F}(\sigma)} \circ C^n(f, \phi, \kappa) \circ \mathrm{incl}_{\mathcal{G}(\rho)} = \begin{cases} \kappa(\sigma) \phi_\sigma & \text{if } \rho = f(\sigma) \\ 0 & \text{otherwise} \end{cases}$$

Proof that $C^n(f, \phi, \kappa)$ assemble to cochain map $C^\bullet(L, \mathcal{G}, s) \to C^\bullet(K, \mathcal{F}, r)$. We need to prove that the following square commutes

$$\begin{array}{ccc} C^n(K, \mathcal{F}) & \xrightarrow{d^n_{K,\mathcal{F},r}} & C^{n+1}(K, \mathcal{F}) \\ {\scriptstyle C^n(f,\phi,\kappa)} \uparrow & & \uparrow {\scriptstyle C^{n+1}(f,\phi,\kappa)} \\ C^n(L, \mathcal{G}) & \xrightarrow[d^n_{L,\mathcal{G},s}]{} & C^{n+1}(L, \mathcal{G}) \end{array}$$

The previous square commutes if and only if the two paths sandwiched between an inclusion and projection are equal, as the two paths are maps between a coproduct and a product. Evaluating the first path,

$$\begin{aligned} \mathrm{proj}_{\mathcal{F}(\sigma)} \circ d^n_{K,\mathcal{F},r} \circ C^n(f, \phi, \kappa) \circ \mathrm{incl}_{\mathcal{G}(\rho)} &= \sum_{\sigma' \in S_K^n \mid f(\sigma') = \rho \wedge \sigma' \subseteq \sigma} \kappa(\sigma') r(\sigma', \sigma) \mathcal{F}(\sigma' \subseteq \sigma) \circ \phi_{\sigma'} \\ &= \sum_{\sigma' \in S_K^n \mid f(\sigma') = \rho \wedge \sigma' \subseteq \sigma} \kappa(\sigma') r(\sigma', \sigma) \phi_\sigma \circ \mathcal{G}(f(\sigma') \subseteq f(\sigma)) \\ &= \left(\sum_{\sigma' \in S_K^n \mid f(\sigma') = \rho \wedge \sigma' \subseteq \sigma} \kappa(\sigma') r(\sigma', \sigma) \right) \phi_\sigma \circ \mathcal{G}(\rho \subseteq f(\sigma)) \end{aligned}$$

where in the second line, we have used that naturality of $\phi \colon \mathcal{G} \circ i_{\mathrm{PC}}(S_f) \to \mathcal{F}$. Now, we evaluate the second path,

$$\begin{aligned} \mathrm{proj}_{\mathcal{F}(\sigma)} \circ C^{n+1}(f, \phi, \kappa) \circ d^n_{L,\mathcal{G},s} \circ \mathrm{incl}_{\mathcal{G}(\rho)} &= \sum_{\rho' \in S_L^{n+1} \mid f(\sigma) = \rho' \wedge \rho \subseteq \rho'} \kappa(\sigma) s(\rho, \rho') \phi_\sigma \circ \mathcal{G}(\rho \subseteq \rho') \\ &= \begin{cases} \kappa(\sigma) s(\rho, f(\sigma)) \phi_\sigma \circ \mathcal{G}(\rho \subseteq f(\sigma)) & \text{if } f(\sigma) \in S_L^{n+1} \wedge \rho \subseteq f(\sigma) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

This equals the previous expression by using Lemma 1.2, and thus the square commutes. $\square$

Definition 2.3. We define the functor $C^\bullet \colon \mathrm{Sh}_{\mathcal{A}}\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}^{\mathrm{op}} \to \mathrm{CoCh}^{\geq 0}(\mathcal{A})$ by the previous constructions.

Proof that $C^\bullet$ is a functor. We need the check if $C^\bullet$ preserrves identites and compositions.

Identity Let $\sigma, \rho \in S_K^n$ be any two simplicies. Then

$$\mathrm{proj}_{\mathcal{F}(\sigma)} \circ C^n(\mathrm{id}_K, \mathrm{id}_{\mathcal{F}}, c_1) \circ \mathrm{incl}_{\mathcal{F}(\rho)} = \begin{cases} 1 \cdot \mathrm{id}_{\mathcal{F}(\sigma)} & \text{if } \rho = \sigma \\ 0 & \text{otherwise} \end{cases}$$

This is precisely the identity when $\rho = \sigma$ and 0 otherwise. These are the components of

$$\mathrm{id}_{C^n(K,\mathcal{F},r)} \colon \bigoplus_{\sigma \in S_K^n} \mathcal{F}(\sigma) \to \bigoplus_{\sigma \in S_K^n} \mathcal{F}(\sigma)$$

Hence $C^n(\mathrm{id}_{K,\mathcal{F},r}) = \mathrm{id}_{C^n(K,\mathcal{F},r)}$.

Composition Let $(f,\phi,\kappa) \colon (K,\mathcal{F},r) \to (L,\mathcal{G},s)$ and $(g,\psi,\lambda) \colon (L,\mathcal{G},s) \to (H,\mathcal{I},t)$ be morphisms. For any two simplices $\sigma \in S_K^n$ and $\mu \in S_H^n$, we compute

$$\begin{aligned} \mathrm{proj}_{\mathcal{F}(\sigma)} \circ C^n(f,\phi,\kappa) \circ C^n(g,\psi,\lambda) \circ \mathrm{incl}_{\mathcal{I}(\mu)} &= \sum_{\rho \in S_L^n} \begin{cases} \kappa(\sigma)\,\phi_\sigma & \text{if } f(\sigma) = \rho \\ 0 & \text{otherwise} \end{cases} \circ \begin{cases} \lambda(\rho)\,\psi_\rho & \text{if } g(\rho) = \mu \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \kappa(\sigma)\,\lambda(f(\sigma))\,\phi_\sigma \circ \psi_{f(\sigma)} & \text{if } g(f(\sigma)) = \mu \\ 0 & \text{otherwise} \end{cases} \\ &= \mathrm{proj}_{\mathcal{F}(\sigma)} \circ C^n((g,\psi,\lambda) \circ ((f,\phi,\kappa)) \circ \mathrm{incl}_{\mathcal{I}(\mu)} \end{aligned}$$

Thus $C^n(f,\phi,\kappa) \circ C^n(g,\psi,\lambda) = C^n((g,\psi,\lambda) \circ ((f,\phi,\kappa))$.

$\square$

Now, if $\mathcal{A}$ is an abelian category, we have functors $H^n \colon \mathrm{CoCh}^{\geq 0}(\mathcal{A}) \to \mathcal{A}$ (defined in the usual way). Then we define the sheaf cohomology to be the functor

$$\mathrm{Sh}_{\mathcal{A}}\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{C^\bullet} \mathrm{CoCh}^{\geq 0}(\mathcal{A}) \xrightarrow{H^n} \mathcal{A}$$

We often abuse the notation, and denote by H^n the composite.

Remark. If A is abelian and R-enriched, the rig R must be a ring as in an abelian category, the CMon-enrichment as a semiadditive category is extended to an Ab-enrichement.

Remark. Since if $\mathcal{A}$ is abelian, so is $\mathcal{A}^{\mathrm{op}}$, we can play the while game by using $\mathcal{A}^{\mathrm{op}}$ instead. A cosheaf is a functor $S_K \to \mathcal{A}^{\mathrm{op}}$. And in the end H^n lands in $\mathcal{A}^{\mathrm{op}}$. So we have a functor from $\mathrm{Sh}_{\mathcal{A}^{\mathrm{op}}}\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}^{\mathrm{op}} = \mathrm{CoSh}_{\mathcal{A}}\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}}^{\mathrm{op}}$ to $\mathcal{A}^{op}$, or taking opposite categories, a functor (which we denote H_n)

$$H_n \colon \mathrm{CoSh}_{\mathcal{A}}\mathrm{IR}_R\mathrm{SC}_{\mathrm{fin}} \to \mathcal{A}$$

This is called cosheaf homology.

If we take $R = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}-\mathrm{Mod} = \mathrm{Vect}_{\mathbb{Z}/2\mathbb{Z}}$, then consider

$$\begin{tikzcd} \mathrm{Vect}_{\mathbb{Z}/2\mathbb{Z}} & & \mathrm{IR}^{\{1\}}_{\mathbb{Z}/2\mathbb{Z}}\mathrm{SC}_{\mathrm{fin}} \arrow[d, "\text{inclusion}"] \\ & \mathrm{CoSh}_{\mathrm{Vect}_{\mathbb{Z}/2\mathbb{Z}}}\mathrm{IR}_{\mathbb{Z}/2\mathbb{Z}}\mathrm{SC}_{\mathrm{fin}} \arrow[ul, "H_n"] \arrow[r, "p_{\mathrm{CoSh}}"] \arrow[d, "p_{\mathrm{IR}}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{IR}_{\mathbb{Z}/2\mathbb{Z}}\mathrm{SC}_{\mathrm{fin}} \arrow[d, "U_{\mathrm{IR}}"] \\ & \mathrm{CoSh}_{\mathrm{Vect}_{\mathbb{Z}/2\mathbb{Z}}}\mathrm{SC}_{\mathrm{fin}} \arrow[r, "U_{\mathrm{CoSh}}"'] & \mathrm{SC}_{\mathrm{fin}} \arrow[l, bend left, "\Delta(-,\mathbb{Z}/2\mathbb{Z})"] \arrow[ul, dotted, "u"] \arrow[uu, bend right, "A_{\mathrm{IR}}"'] \end{tikzcd}$$

The dotted functor $u \colon \mathrm{SC}_{\mathrm{fin}} \to \mathrm{CoSh}_{\mathrm{Vect}_{\mathbb{Z}/2\mathbb{Z}}}\mathrm{IR}_{\mathbb{Z}/2\mathbb{Z}}\mathrm{SC}_{\mathrm{fin}}$ is induced by the cone induced by $\mathrm{SC}_{\mathrm{fin}}$ over the pullback diagram. Then $H_n \circ u$ is just the usual homology functor (with coefficients in $\mathbb{Z}/2\mathbb{Z}$). We can similarly recover ordinary homology (with coefficients in any ring) for ordered complexes.

3 Discrete Morse Theory

In this section, we try to understand the functorial properties of all constructions in [CGN16].

Definition 3.1. Let $\mathcal{A}$ be an abelian category. A parameterisation of a chain complex $(A^\bullet, d_A^\bullet)$ by a finite graded poset $(X, \leq, \dim)$ is an assignment F such that

$$\begin{aligned} x \in X \mapsto & F(x) \in \mathcal{A} \\ x \prec x' \mapsto & F_{xx'} \colon F(x) \to F(x') \end{aligned}$$

and a collection of isomorphisms $\alpha_n \colon A^n \to \bigoplus_{\dim x = n} F(x)$ such that for any $x, x' \in X$ such that $\dim x = n$ and $\dim x' = n+1$

$$\mathrm{proj}_{F(x')} \circ \alpha_{n+1} \circ d_A^n \circ \alpha_n^{-1} \circ \mathrm{incl}_{F(x)} = \begin{cases} F_{xx'} & \text{if } x \prec x' \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.2. We define the category $\mathrm{GrPos_{fin}CoCh}^{\geq 0}(\mathcal{A})$ as

- Objects: $(X, A^\bullet, F, \alpha)$ such that (F, α) is an X-parameterisation of $A^\bullet$
- A morphism $(X, A^\bullet, F, \alpha) \to (Y, B^\bullet, G, \beta)$ is a tuple (f, a, Φ) where $f \colon X \to Y$ is a morphism of graded posets, $a \colon B^\bullet \to A^\bullet$ is a cochain map and $\Phi = \{\Phi_x \colon G(f(x)) \to F(x)\}_{x \in X}$ is a collection of morphisms such that for any $x \in X$ and $y \in Y$ such that $\dim x = \dim y = n$, we have

$$\mathrm{proj}_{F(x)} \circ \alpha_n \circ a_n \circ \beta_n^{-1} \circ \mathrm{incl}_{G(y)} = \begin{cases} \Phi_x & \text{if } f(x) = y \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.3. $\Pi \colon \mathrm{Sh}_{\mathcal{A}}\mathrm{IR}_R\mathrm{SC_{fin}} \to \mathrm{GrPos_{fin}CoCh}^{\geq 0}(\mathcal{A})$ defined on objects by

$$\begin{aligned} \Pi(K, \mathcal{F}, r) = & (S_K, C^\bullet(K, \mathcal{F}, r), F, \alpha) \\ \text{where } F(\sigma) = & \mathcal{F}(\sigma) \text{ and } F(x \prec x') = r(\sigma, \sigma') \mathcal{F}(\sigma \subseteq \sigma') \text{ and } \alpha_n = \mathrm{id}_{C^n(K, \mathcal{F}, r)} \end{aligned}$$

and given a morphism $(f, \phi, \kappa) \colon (K, \mathcal{F}, r) \to (L, \mathcal{G}, s)$, we define

$$\begin{aligned} \Pi(f, \phi, \kappa) = & (S_f, C^\bullet(f, \phi, \kappa), \Phi) \\ \text{where } \Phi_\sigma = & \kappa(\sigma) \phi_\sigma \end{aligned}$$

It is very easy to see that this preserves identites and composition.

Definition 3.4. We define a category $\mathrm{GrPos_{fin}Par}_{\mathcal{A}}$ as

- Objects: Pairs (X, F) such that X is a graded poset and F is an assignment such that

$$\begin{aligned} x \in X \mapsto & F(x) \in \mathcal{A} \\ x \prec x' \mapsto & F_{xx'} \colon F(x) \to F(x') \end{aligned}$$

  satisfying for any $x, x'' \in X$ such that $\dim x'' = \dim x + 2$, we have

$$\sum_{x' \in X \mid \dim x' = \dim x + 1 \wedge x \prec x' \prec x''} F_{x'x''} \circ F_{xx'} = 0$$

- A morphism $(X, F) \to (Y, G)$ is a pair (f, Φ) where $f \colon X \to Y$ is a map of graded posets and $\Phi = \{\Phi_x \colon G(f(x)) \to F(x)\}_{x \in X}$ is a collection of morphisms such that for any $x' \in X$ and $y \in Y$ such that $\dim x' = n+1$ and $\dim y = n$, we have

$$\sum_{x \in X \mid \dim x = n \wedge x \prec x' \wedge f(x) = y} F_{xx'} \circ \Phi_x = \begin{cases} \Phi_{x'} \circ G_{yf(x')} & \text{if } \dim f(x') = n+1 \wedge y \prec f(x') \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.5. Define a functor $\mathrm{AC}\colon \mathrm{GrPos_{fin}Par}_{\mathcal{A}} \to \mathrm{GrPos_{fin}CoCh}^{\geq 0}(\mathcal{A})$ which maps (X, F) to $(X, A^{\bullet}, F, \alpha)$ where $A^n = \bigoplus_{x \in X \mid \dim x = n} F(x)$ and the differential is given in the usual way and each α_n is identity.

Lemma 3.1. AC is an equivalence of categories.

Thus from now on, we will work with the category $\mathrm{GrPos_{fin}Par}_{\mathcal{A}}$.

Definition 3.6. Given an object $(X, F) \in \mathrm{GrPos_{fin}Par}_{\mathcal{A}}$, an F-compatible acyclic partial matching on X is a subset $\Sigma \in X \times X$ such that

- If $(x, x') \in \Sigma$ then $x \prec x'$
- If $(x, x') \in \Sigma$, then neither x nor x' appears in any other pair in Σ
- If $(x, x') \in \Sigma$, then $F_{xx'}$ is an isomorphism
- Given two pairs $(x_1, x'_1), (x_2, x'_2) \in \Sigma$, we say $(x_1, x'_1) \lhd (x_2, x'_2)$ if and only if $x_1 \prec x'_2$. We require the transitive closure of the relation $\lhd$ to give be a poset on Σ.

Definition 3.7. Let Σ be an F-compatible acyclic partial matching on X.

- A gradient path γ in Σ is a strictly $\lhd$-increasing sequence of elements of Σ, i,e,

$$\gamma = (x_1, x'_1) \lhd \cdots \lhd (x_n, x'_n) \text{ where } n > 0$$

  For such a path, $\mathrm{s}_\gamma = x'_1$ and $\mathrm{t}_\gamma = x_n$.

- Define $M_{(X,\Sigma)} = X \setminus (\mathrm{proj}_l(\Sigma) \cup \mathrm{proj}_r(\Sigma))$ where $\mathrm{proj}_l, \mathrm{proj}_r\colon X \times X \to X$ are the left and right projections respectively. For $m, m' \in M_{(X,\Sigma)}$, we say a path γ flows from m to m' if $m \prec \mathrm{s}_\gamma$ and $\mathrm{t}_\gamma \prec m'$. We define the relation $\leq_\Sigma \subseteq M_{(X,\Sigma)} \times M_{(X,\Sigma)}$ as

$$\text{transitive closure } \{m, m' \text{ are related if } \exists \text{ gradient path flowing from } m \text{ to } m' \text{ or if } m \leq m' \text{ in } X\}.$$

  Also, let the dimension of elements of $M_{(X,\Sigma)}$ just be the dimension as elements of X.

- For a gradient path $\gamma = (x_1, x'_1) \lhd \cdots \lhd (x_n, x'_n)$, define $F_\gamma\colon F(x'_1) \to F(x_n)$ as

$$F_\gamma = \left(-F^{-1}_{x_n x'_n}\right) \circ F_{x_{n-1}x'_n} \circ \cdots \circ F_{x_1 x'_2} \circ \left(-F^{-1}_{x_1 x'_1}\right)$$

- Now define an assignment on $M_{(X,\Sigma)}$ denoted by F^Σ which assigns

$$\begin{aligned} F^\Sigma(m) &= F(m) \\ F^\Sigma_{mm'} &= F_{mm'} + \sum_{\gamma \mid m \prec \mathrm{s}_\gamma \wedge \mathrm{t}_\gamma \prec m'} F_{\mathrm{t}_\gamma m'} \circ F_\gamma \circ F_{m \mathrm{s}_\gamma} \end{aligned}$$

Lemma 3.2. $\left(M_{(X,\Sigma)}, F^\Sigma\right)$ is an object of $\mathrm{GrPos_{fin}Par}_{\mathcal{A}}$.

Proof. [CGN16, Proposition 3.1] □

Definition 3.8. We define a category $\mathrm{APM}_{\mathcal{A}}$ as:

- Objects: triples (X, F, Σ) such that $(X, F) \in \mathrm{GrPos_{fin}Par}_{\mathcal{A}}$ and Σ is an F-compatible acyclic partial matching on X.
- A morphism $(X, F, \Sigma) \to (Y, G, \Gamma)$ is just a morphism $(f, \Phi)\colon (X, F) \to (Y, G)$ such that for any $m \in M_{(X,\Sigma)}$, we have $f(m) \in M_{(Y,\Gamma)}$

3.1 Only one non-critial element

Let $(X, \mathcal{F}, \Sigma)$ and $(Y, \mathcal{G}, \Pi)$ be graded posets with assignments and compatible acyclic partical matchings (with only one critical element). If we want the assignment of Morse data to be a functor, we need an induced map from $M_{(X,\Sigma)}$ to $M_{(X,\Sigma)}$. A morphism $(f, \phi) \colon (X, \mathcal{F}, \Sigma) \to (Y, \mathcal{G}, \Pi)$ has to satisfy

$$f\left(X \setminus \{x_1, x_1'\}\right) \subseteq Y \setminus \{y_1, y_1'\}$$

Let $\Sigma = \{(x_1, x_1')\}$ and $\Pi = \{(y_1, y_1')\}$. The poset on $M_{X,\Sigma}$ is given by

$$\text{transitive closure of } \{m \leq m' \vee m \prec x_1' \succ x_1 \prec m'\}$$

This is given by $m \leq m' \vee m \leq x_1' \succ x_1 \leq m'$. Thus if f were to induce a monotone map on the Morse data, we would need

$$m \leq m' \vee m \leq x_1' \succ x_1 \leq m' \implies f(m) \leq f(m') \vee f(m) \leq y_1' \succ y_1 \leq f(m')$$

Since f is monotone, we already have $m \leq m' \implies f(m) \leq f(m')$. So The second condition will be true (in general) iff $f(x_1) = y_1$ and $f(x_1') = y_1'$. Also, Φ will have to satisfy the conditions that Φ_{x_1} and $\Phi_{x_1'}$ are iso to get a map between the assignments (and hence a chain map between the Morse data).

I am not sure how to generalise these conditions to the case of more non-critical elements, as these conditions seem way too restrictive.This will help define the category $\mathrm{APA}_{\mathcal{A}}$.

A Finite simplicial complexes with an incidence relation form a category

Proof that composition is of morphisms is a morphism. Given two morphisms $(f, \kappa) \colon (K, r) \to (L, s)$ and $(g, \lambda) \colon (L, s) \to (H, t)$, we want to show that the composite (h, ω) defined by $h = g \circ f$ and $\omega(\sigma) = \kappa(\sigma)\lambda(f(\sigma))$ for any $\sigma \in S_K$ is a morphism from $(K, r) \to (H, t)$. So we choose $\sigma \in S_K^{n+1}$ and $\mu \in S_H^n$ and we compute

$$\begin{aligned}
\sum_{\sigma' \in S_K^n | h(\sigma') = \mu \wedge \sigma' \subseteq \sigma} \omega(\sigma') r(\sigma', \sigma) &= \sum_{\rho \in S_L^n | g(\rho) = \mu \wedge \rho \subseteq f(\sigma)} \left(\sum_{\sigma' \in S_K^n | f(\sigma') = \rho \wedge \sigma' \subseteq \sigma} \kappa(\sigma') \lambda(f(\sigma')) r(\sigma', \sigma) \right) \\
&= \sum_{\rho \in S_L^n | g(\rho) = \mu \wedge \rho \subseteq f(\sigma)} \lambda(\rho) \left(\sum_{\sigma' \in S_K^n | f(\sigma') = \rho \wedge \sigma' \subseteq \sigma} \kappa(\sigma') r(\sigma', \sigma) \right) \\
&= \sum_{\rho \in S_L^n | g(\rho) = \mu \wedge \rho \subseteq f(\sigma)} \lambda(\rho) \begin{cases} \kappa(\sigma) s(\rho, f(\sigma)) & \text{if } f(\sigma) \in S_L^{n+1} \wedge \rho \subseteq f(\sigma) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \kappa(\sigma) \sum_{\rho \in S_L^n | g(\rho) = \mu \wedge \rho \subseteq f(\sigma)} \lambda(\rho) s(\rho, f(\sigma)) & \text{if } f(\sigma) \in S_L^{n+1} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \kappa(\sigma) \begin{cases} \lambda(f(\sigma)) t(\mu, g(f(\sigma))) & \text{if } g(f(\sigma)) \in S_H^{n+1} \wedge \mu \subseteq g(f(\sigma)) \\ 0 & \text{otherwise} \end{cases} & \text{if } f(\sigma) \in S_L^{n+1} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \kappa(\sigma) \lambda(f(\sigma)) t(\mu, g(f(\sigma))) & \text{if } g(f(\sigma)) \in S_H^{n+1} \wedge \mu \subseteq g(f(\sigma)) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \omega(\sigma) t(\mu, h(\sigma)) & \text{if } h(\sigma) \in S_H^{n+1} \wedge \mu \subseteq h(\sigma) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

□

Composition is associative. Given three morphisms $(f,\kappa): (K,r) \to (L,s)$, $(g,\lambda): (L,s) \to (H,t)$ and $(h,\theta): (H,t) \to (I,u)$, we want to show that the composition is associative.

$$\begin{aligned}
&((h,\theta)\circ(g,\lambda))\circ(f,\kappa) = (h\circ g,\omega)\circ(f,\kappa) \text{ where } \omega\colon S_L \to R \text{ is given by } \omega(\rho) = \lambda(\rho)\,\theta(g(\rho)) \\
&= (h\circ g\circ f,\omega') \text{ where } \omega'\colon S_K \to R \text{ is given by } \omega'(\sigma) = \kappa(\sigma)\,\omega(f(\sigma)) = \kappa(\sigma)\,\lambda(f(\sigma))\,\theta(g(f(\sigma)))
\end{aligned}$$

$$\begin{aligned}
&(h,\theta)\circ((g,\lambda)\circ(f,\kappa)) = (h,\theta)\circ(g\circ f,\omega'') \text{ where } \omega''\colon S_K \to R \text{ is given by } \omega''(\sigma) = \kappa(\sigma)\,\lambda(f(\sigma)) \\
&= (h\circ g\circ f,\omega''') \text{ where } \omega'''\colon S_K \to R \text{ is given by } \omega'''(\sigma) = \omega''(\sigma)\,\theta(g(f(\sigma))) = \kappa(\sigma)\,\lambda(f(\sigma))\,\theta(g(f(\sigma)))
\end{aligned}$$

Since we know composition of morphisms of simplicial complexes is associative, we do not distinguish between $h\circ(g\circ f)$ and $(h\circ g)\circ f$. Since $\omega' = \omega'''$, we see that the second component of the compositions are also equal, and hence the composition is associative. □

Proof that identities exist. Let (K,r) be an object. Consider the morphism (id_K, c_1) where $c_1\colon S_K \to R$ maps everything to $1 \in R$. To show that this is a morphism, we need to show that for any $\rho \in S_K^n$ and any $\sigma \in S_K^{n+1}$, we have

$$\sum_{\sigma'\in S_K^n \mid \sigma'=\rho\wedge\sigma'\subseteq\sigma} r(\sigma',\sigma) = \begin{cases} s(\rho,f(\sigma)) & \text{if } \rho\subseteq\sigma \\ 0 & \text{otherwise} \end{cases}$$

This is obviously true. Now let $(f,\kappa): (K,r) \to (L,s)$ be a morphism.

$$\begin{aligned}
&(f,\kappa)\circ(\mathrm{id}_K,c_1) = (f,\omega) \text{ where } \omega\colon S_K \to R \text{ is given by } \omega(\sigma) = 1\cdot\kappa(\sigma)\text{, so } \omega=\kappa \\
&(\mathrm{id}_L,c_1)\circ(f,\kappa) = (f,\omega') \text{ where } \omega'\colon S_K \to R \text{ is given by } \omega'(\sigma) = \kappa(\sigma)\cdot 1\text{, so } \omega=\kappa
\end{aligned}$$

Thus $(\mathrm{id}_K, c_1) = \mathrm{id}_{(K,r)}$. □

References

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