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KNOT INVARIANTS FROM GAUGE THEORY

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Abstract. — The work presented here appears at the crossroads of mathematics and physics and rests primarily on the pioneering ideas of Vaughan Jones and Edward Witten [Wit89] in establishing ground-breaking connections between gauge theory (specifically the topological Chern-Simons theory) to Knot invariants and Invariants of 3-manifolds in general. The standard texts in mathematics were referred to [Lee12], [Mun00] for providing basic definitions and structure for Topology and Differential Geometry. Online published lecture notes [Bö9] on Gauge theory aimed at developing the mathematical underpinnings of gauge theory using bundle theory were extensively referred for the flow. Other references include the review article [CDM11] on modern methods in Knot theory, lecture notes based on a talk [Liu14] on Introduction to Chern-Simons theory, a thesis [P.R96] very closely related to the current topic.

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CHAPTER 1

MATHEMATICAL STRUCTURE OF GAUGE THEORY

1.1. Smooth Fibre Bundles

Definition 1.1.1. — Let E, B, F be manifolds, let $\pi: E \rightarrow B$ be a surjective smooth map. (E, B, π, F) is a **smooth fibre bundle** if each point $x \in B$ has an open neighbourhood $U \subseteq B$ such that there exists a diffeomorphism $\psi_U: \pi^{-1}(U) \xrightarrow{\sim} U \times F$ (called a **local trivialisation**) such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\sim]{\psi_U} & U \times F \\ \pi|_{\pi^{-1}(U)} \downarrow & \nearrow \text{proj}_1 & \\ U & & \end{array}$$

B is called the **base space**, E is called the **étalé space** and F is called the **characteristic fibre**. ψ_U is the **local trivialization over U** . We also say that E is a **bundle over B** . For $b \in B$, we call $\pi^{-1}(b)$ the **fibre above b** .

Definition 1.1.2. — A smooth map $s: U \rightarrow E$ such that

$$\begin{array}{ccc} & & E \\ & \nearrow s & \downarrow \pi \\ U & \xrightarrow{\iota} & B \end{array}$$

commutes is called a **local section** of the bundle over U . The set of local sections is written as $\Gamma_E U$. A local section is called a **global section** if $U = B$.

Definition 1.1.3. — A **map of bundles** is defined as two maps $f: E_1 \rightarrow E_2$ and $f_B: B_1 \rightarrow B_2$ satisfying

$$\begin{array}{ccc} E_1 & \xrightarrow{f_E} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ B_1 & \xrightarrow{f_B} & B_2 \end{array}$$

A **map of bundles over B** is the same as above with $f_B = \text{id}_B$. The commuting of the square ensures that the fibre above $b \in B_1$ is mapped to the fibre above $f_B(b) \in B_2$.

Lemma 1.1.4. — Let $\pi_2: E_2 \rightarrow B_2$ be a bundle and a map $f_B: B_1 \rightarrow B_2$, the pullback $E_1 = E_2 \times_{B_1} B_2$ can be given bundle structure.

Proof. — We construct the pullback bundle (E_1, B_1, π_1, F_1) as $E_1 = \{(b, e) \in B_1 \times E_2 \mid f_B(b) = \pi_2(e)\}$, $\pi_1 = \text{proj}_1: E_1 \rightarrow B_1$ and $F_1 = F_2$. To construct a local trivialisation around $b \in B_1$, choose a local trivialisation around $f_B(b)$. Let this be $\psi_{U_2}: \pi_2^{-1}(U_2) \rightarrow U_2 \times F_2$. Let $U_1 = f_B^{-1}(U_2)$. Then,

$$\begin{aligned} \pi_1^{-1}(U_1) &= \text{proj}_1^{-1}(U_1) \\ &= \{(b_1, e_2) \in U_1 \times E_2 \mid f_B(b_1) = \pi_2(e_2)\} \\ &\cong \{(b_1, b_2, f_2) \in U_1 \times U_2 \times F_2 \mid f_B(b_1) = b_2\} \cong U_1 \times F_2 \end{aligned}$$

□

Lemma 1.1.5 (Local existence of lifts). — Let (E, B, π, F) be a smooth fibre bundle. For any smooth curve $\gamma: I \rightarrow B$ and a $t_0 \in I$ we have a connected open subset of I called $I_{t_0} \ni t_0$ such that for any $u_0 \in \pi^{-1}(\gamma(t_0))$ we have a smooth map $\tilde{\gamma}_{t_0, u_0}: I_{t_0} \rightarrow E$ such that

- $\tilde{\gamma}_{t_0, u_0}(t_0) = u_0$
- $\pi \circ \tilde{\gamma}_{t_0, u_0} = \gamma|_{I_{t_0}}$ (*Lift*)

Proof. — Let $\gamma: I \rightarrow B$ be a smooth path and $t_0 \in I$. Let $U \ni t_0$ be an open set over which the bundle is trivial with trivialisation $\psi_U: \pi^{-1}(U) \rightarrow U \times F$. Then $\gamma^{-1}(U)$ is open in I by continuity. Let I_{t_0} be the connected component of $\gamma^{-1}(U)$ which contain t_0 . Obviously, I_{t_0} is open, connected and contains t_0 . Let $u_0 \in \pi^{-1}(\{\gamma(t_0)\})$ and let $\psi(u_0) = (\gamma(t_0), f_0)$. Define $\tilde{\gamma}_{t_0, u_0}: I_{t_0} \rightarrow E$ as $\tilde{\gamma}_{t_0, u_0}(t) = \psi^{-1}(\gamma(t), f_0)$. Now, for $t \in I_{t_0}$ we have

$$\pi(\tilde{\gamma}_{t_0, u_0}(t)) = \pi(\psi^{-1}(\gamma(t), f_0)) = \text{proj}_1(\gamma(t), f_0) = \gamma(t)$$

So, we have the lift condition. Almost by definition, $\tilde{\gamma}_{t_0, u_0}(t_0) = u_0$. It is also obvious that $\tilde{\gamma}_{t_0, u_0}$ is smooth. □

Definition 1.1.6. — Let (E, B, π, F) be a smooth fibre bundle. For each $u \in E$, let V_u denote the subspace of $T_u E$ consisting of vectors which are tangent to the fibre at $\pi(u)$, i.e. $V_u = \ker(d\pi|_u)$. We call V_u the **vertical subspace** at u . An **Ehressmann connection** on $\pi: E \rightarrow B$ is distribution $u \mapsto H_u$ of vector subspaces $H_u \subset T_u E$, called **horizontal subspaces**, such that:

- $H_u \cap V_u = 0$
- for each $u \in E$ we have $T_u E = H_u \oplus V_u$ and (complementarity).
- the assignment $u \mapsto H_u$ depends smoothly on $u \in E$, this means that in the unique decomposition of any smooth vector field X on $U \subseteq E$ open as

$$X = X_H + X_V$$

such that $X_H(u) \in H_u$ and $X_V(u) \in V_u$, we have that X_H is smooth (or equivalently X_V is smooth, or equivalently both) as a section of TE .

Lemma 1.1.7. — Given a smooth fibre bundle (E, B, π, F) with an Ehressmann connection $u \mapsto H_u$, we have a canonical isomorphism $\phi_u: H_u \xrightarrow{\sim} T_{\pi(u)} B$

Proof. — We have $V_u = \ker(d\pi|_u)$ and $T_u E = V_u \oplus H_u$. Since π is surjective, $\text{im}(d\pi|_u) = T_{\pi(u)} B$. So, we have $\phi_u: T_u E/V_u = H_u \xrightarrow{\sim} T_{\pi(u)} B$ from the first isomorphism theorem ($G/\ker(\phi) \simeq \text{im}(\phi)$). $\square$

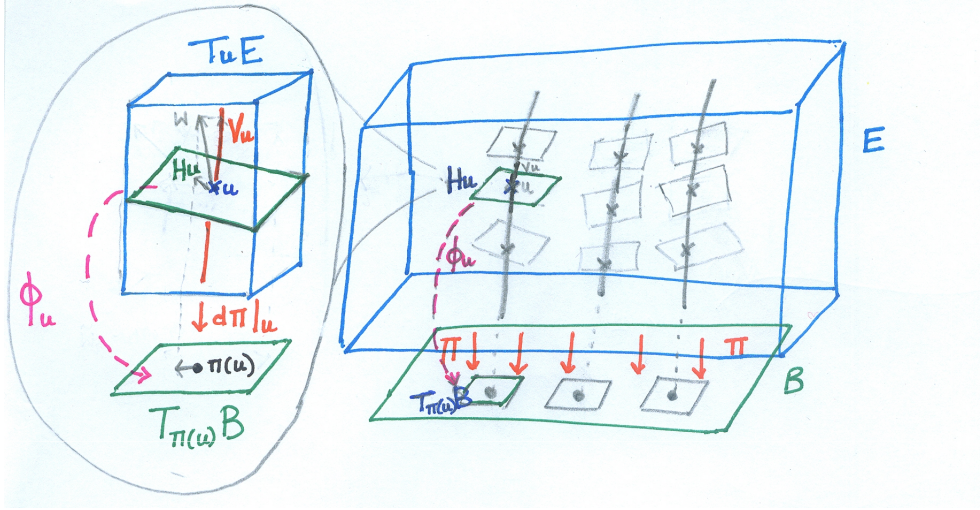


FIGURE 1. A fibre bundle with an Ehressmann connection showing the horizontal subspaces as small linearised neighbourhoods of points

1.2. Principal G -Bundles

Definition 1.2.1. — Let us have a smooth fibre bundle (E, B, π, F) , a Lie-group G , a differentiable right-group action of G on E given by $a: E \times G \rightarrow E$ (denoted as $a(p, g) = p \cdot g = a_g(p)$). These together form a **principal G -bundle** if:

- G preserves the fibres on E , i.e. for $x \in B$ let F_x denote $\pi^{-1}(x)$, the fibre on x , if $y \in F_x$, then $y \cdot g \in F_x \forall g \in G$
- The action of $G \curvearrowright E$ is *free* and *transitive* on each fibre.
- $\forall x \in B$ and $\forall y \in F_x$ in their corresponding fibres, the map $G \rightarrow F_x$ that sends $g \mapsto y \cdot g$ is a diffeomorphism, i.e. each fibre of $E \rightarrow B$ looks like G .
- The above condition shows us that $F \cong G$. We will require the existence of a local trivialisation of the bundle around every point $p \in B$ ($U, \psi_U: \pi^{-1}(U) \rightarrow U \times G$) such that for if $\psi_U(u) = (\pi(u), g)$, then

$$\psi_U(u \cdot h) = (\pi(u), gh)$$

That is, the action looks like right multiplication locally.

The first three conditions can be alternatively expressed as, the canonical morphism $(\text{proj}_1, a): E \times G \rightarrow E \times_B E$ is an isomorphism.

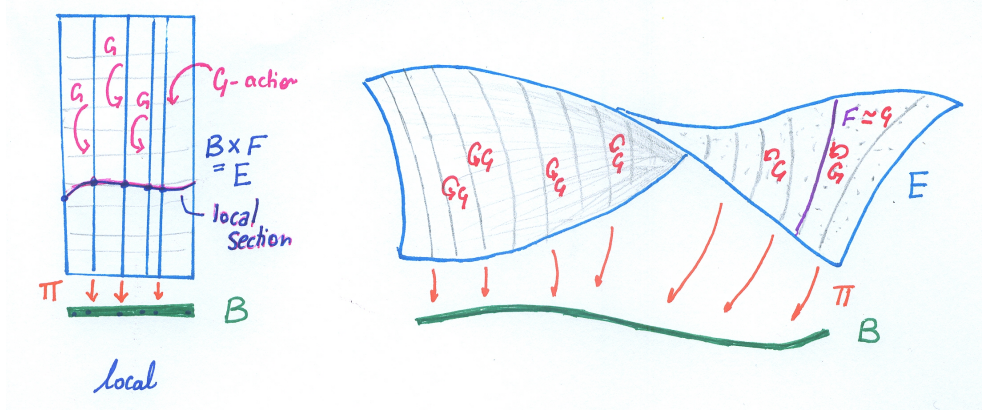
$$\begin{array}{ccccc}
 E \times G & & \xrightarrow{\quad a \quad} & & E \\
 \downarrow \text{proj}_1 & \searrow \exists! & & \searrow p_1 & \downarrow \pi \\
 & E \times E & \xrightarrow{\quad p_1 \quad} & E & \\
 & \downarrow p_2 & \lrcorner & & \\
 & E & \xrightarrow{\quad \pi \quad} & B &
 \end{array}$$

Also, it is easy to see that $E/G \cong B$. A G -bundle that satisfies the fourth condition globally is called **trivial**.

Definition 1.2.2. — Let (E, B, π, G) and (E', B', π', G) be G -principal bundles. A map $f: E \rightarrow E'$ is a **morphism of principal G -bundles** if it is compatible with the action of G , i.e.

$$\forall p \in E, g \in G \quad [f(p \cdot g) = f(p) \cdot g]$$

This implies that $f: E \rightarrow E'$ gives rise to a morphism of bases spaces as it preserves the fibre structure (maps fibres to fibres). Let this induced map $f_B: B \rightarrow B'$. If f is a diffeomorphism $E \rightarrow E'$ and a morphism of G -bundles, then we call it an **automorphism**. Now, an automorphism of E that induces id_B on its base space is called a **gauge transformation**.

FIGURE 2. Illustration of a principal G -bundle and a local section

Definition 1.2.3. — We call a Ehressmann connection on a principal G -bundle a G -equivariant or a **principal connection** if

$$da_g|_u(H_u) = H_{u \cdot g}$$

Definition 1.2.4. — A **connection form** on a G -bundle is a $\mathfrak{g}$ -valued 1-form ω such that

- For any $X \in \mathfrak{g}$, we have $\omega(X^\#) = X$ (equivalently if $(a_p^*\omega)_e = \text{id}_{\mathfrak{g}}$ for any $p \in E$)
- For any $g \in G$, we have $a_g^*\omega = \text{Ad}_{g^{-1}} \circ \omega$

Given a connection form ω , define a principal connection as $u \mapsto \ker \omega_u$. It is easy to check the properties. We can write $v \in T_u E$ as $v = X_h + X_u^\#$ as the tangent space is the sum of horizontal and vertical spaces and the vertical space consists of fundamental fields evaluated at u . define $\omega_u(v) = X$ where $X \in \mathfrak{g}$ gives $X^\#$. This is again easy to check that this is a connection 1-form.

Lemma 1.2.5. — Let $(\pi: E \rightarrow B, F, a), (\pi': E' \rightarrow B', F', a')$ be principal G -bundles and let $\omega: TE' \rightarrow \mathfrak{g}$ be a connection form and $f: E \rightarrow E'$ be a morphism of principal G -bundles. Then $f^*\omega: TE \rightarrow \mathfrak{g}$ is also a connection form.

Proof. — Let $p \in E$

$$(a_p^*(f^*\omega))_e = ((f \circ a_p)^*)_e = (a_{f(p)}^*\omega)_e = \text{id}_{\mathfrak{g}}$$

Let $g \in G$,

$$\begin{aligned} a_g^*(f^*\omega) &= (f \circ a_g)^*\omega = (a_g \circ f)^*\omega = f^*(a_g^*\omega) \\ &= \text{Ad}_{g^{-1}} \circ \omega \circ df = \text{Ad}_{g^{-1}} \circ f^*\omega \end{aligned}$$

Hence $f^*\omega: TE \rightarrow \mathfrak{g}$ is a connection form. $\square$

Lemma 1.2.6. — *Let $(\pi: E \rightarrow B, F, a)$ be a principle G -bundle with a connection form ω . For any smooth curve $\gamma: I \rightarrow B$, a $t_0 \in I$ and a $u_0 \in \pi^{-1}(\gamma(t_0))$, we have a connected open subset of I called $I_{t_0, u_0} \ni t_0$ and a unique smooth map $\gamma_{t_0, u_0}^h: I_{t_0, u_0} \rightarrow E$ such that*

- $\gamma_{t_0, u_0}^h(t_0) = u_0$
- $\pi \circ \gamma_{t_0, u_0}^h = \gamma|_{I_{t_0, u_0}}$ (*Lift*)
- $(\gamma_{t_0, u_0}^h)^* \omega = 0$ (*Horizontal*)

Note: γ_{t_0, u_0}^h is unique only once we fix the choice of I_{t_0, u_0} . By enlarging and contracting the interval of definition, we will get different horizontal lifts, but by the above lemma, they will have to match on their common domain of definition.

Proof. — Let $\gamma: I \rightarrow B$ be a smooth curve, let $t_0 \in I$ and $u_0 \in \pi^{-1}(\{\gamma(t_0)\})$. Then by 1.1.5 we have an $I_{t_0} \ni t_0$ which is an open, connected subset of I and a smooth map $\tilde{\gamma}_{t_0, u_0}: I_{t_0} \rightarrow E$ such that $\tilde{\gamma}_{t_0, u_0}(t_0) = u_0$ and $\pi \circ \tilde{\gamma}_{t_0, u_0} = \gamma|_{I_{t_0}}$. Now, let $\hat{\gamma}_{t_0, u_0}$ be some other smooth lift of $\gamma|_{I_{t_0}}$ such that $\hat{\gamma}_{t_0, u_0}(t_0) = u_0$. Then by the freeness, transitivity and smoothness of the action, we have a unique smooth (smoothness follows from the second property in the definition of G -bundle) $\hat{g}_{t_0, u_0}: I_{t_0} \rightarrow G$ such that for all $t \in I_{t_0}$ we have $\hat{\gamma}_{t_0, u_0}(t) = \tilde{\gamma}_{t_0, u_0} \cdot \hat{g}_{t_0, u_0}(t)$ and $\hat{g}_{t_0, u_0} = e$.

So let us suppose there is a horizontal lift $\gamma_{t_0, u_0}^h: I_{t_0} \rightarrow E$. Then, by the above argument, there must exist a unique $g^h: I_{t_0} \rightarrow G$ such that $\gamma_{t_0, u_0}^h(t) = \tilde{\gamma}_{t_0, u_0}(t) \cdot g_{t_0, u_0}^h(t)$ for all $t \in I_{t_0}$ and $g_{t_0, u_0}^h = e$. For γ_{t_0, u_0}^h to be horizontal, we must have

$$\begin{aligned} \omega_{\gamma_{t_0, u_0}^h(t)}(\dot{\gamma}_{t_0, u_0}^h(t)) &= 0 \iff \omega_{\gamma_{t_0, u_0}^h(t)} \circ d\gamma_{t_0, u_0}^h|_t = 0 \\ &\iff \omega \circ d\gamma_{t_0, u_0}^h = 0 \iff (\gamma_{t_0, u_0}^h)^* \omega = 0 \end{aligned}$$

Now,

$$\begin{aligned} d\gamma_{t_0, u_0}^h|_t &= da_{\tilde{\gamma}_{t_0, u_0}(t)}|_{g_{t_0, u_0}^h(t)} \circ dg_{u_0, t_0}^h|_t + da_{g_{t_0, u_0}^h(t)}|_{\tilde{\gamma}_{t_0, u_0}(t)} \circ d\tilde{\gamma}_{t_0, u_0}|_t \\ &= da_{\gamma_{t_0, u_0}^h(t)}|_e \circ dL_{g_{t_0, u_0}^h(t)}^{-1}|_{g_{t_0, u_0}^h(t)} \circ dg_{u_0, t_0}^h|_t \\ &\quad + da_{g_{t_0, u_0}^h(t)}|_{\tilde{\gamma}_{t_0, u_0}(t)} \circ d\tilde{\gamma}_{t_0, u_0}|_t \end{aligned}$$

where in the second line we used $a_{\tilde{\gamma}_{t_0, u_0}(t)} = a_{\gamma_{t_0, u_0}^h(t)} \circ L_{g_{u_0, t_0}^h(t)}^{-1}$. Since $\omega_p \circ da_p|_e = \text{id}_{\text{Lie}(G)}$ and $a_h^* \omega = \text{Ad}_{h^{-1}} \circ \omega$, we have

$$\begin{aligned} \omega_{\gamma_{t_0, u_0}^h(t)} \circ d\gamma_{t_0, u_0}^h|_t &= dL_{g_{t_0, u_0}^h(t)}^{-1}|_{g_{t_0, u_0}^h(t)} \circ dg_{u_0, t_0}^h|_t \\ &\quad + \text{Ad}_{g_{t_0, u_0}^h(t)}^{-1} \circ \omega_{\tilde{\gamma}_{t_0, u_0}(t)} \circ d\tilde{\gamma}_{t_0, u_0}|_t \\ &= dL_{g_{t_0, u_0}^h(t)}|_{g_{t_0, u_0}^h(t)} \circ \left(dg_{t_0, u_0}^h|_t + dR_{g_{t_0, u_0}^h(t)}|_e \circ (\tilde{\gamma}_{t_0, u_0}^* \omega)_t \right) \\ &= 0 \end{aligned}$$

Since L_h is a diffeomorphism, dL_h is invertible for any $h \in G$. So we have

$$dg_{t_0, u_0}^h|_t = -dR_{g_{t_0, u_0}^h(t)}|_e \circ (\tilde{\gamma}_{t_0, u_0}^* \omega)_t$$

Choose any chart of G around e . That is, we have an open $V \subseteq G$ and a diffeomorphism $\phi: V \rightarrow \mathbb{R}^n$ such that $e \in V$. Then still assuming $\gamma_{t_0, u_0}^h: I_{t_0} \rightarrow G$ exists (and hence $g_{t_0, u_0}^h: I_{t_0} \rightarrow G$ exists), we define $J_{t_0, V}$ to be the connected component of $g_{t_0, u_0}^{h^{-1}}(V)$. $J_{t_0, V}$ obviously contains t_0 , is open in I_{t_0} (and hence open in I , as I_{t_0} is open in I) and connected. Then, for $t \in J_{t_0, V}$

$$\begin{aligned} dg_{t_0, u_0}^h|_t &= -dR_{g_{t_0, u_0}^h(t)}|_e \circ (\tilde{\gamma}_{t_0, u_0}^* \omega)_t \\ \iff d\phi|_{g_{t_0, u_0}^h(t)} \circ dg_{t_0, u_0}^h|_t &= -d\phi|_{g_{t_0, u_0}^h(t)} \circ dR_{\phi^{-1}(\phi(g_{t_0, u_0}^h(t)))}|_e \circ (\tilde{\gamma}_{t_0, u_0}^* \omega)_t \\ \iff dg_{t_0, u_0}^c|_t &= -d\phi|_{\phi^{-1}(g_{t_0, u_0}^c(t))} \circ dR_{\phi^{-1}(g_{t_0, u_0}^c(t))}|_e \circ (\tilde{\gamma}_{t_0, u_0}^* \omega)_t \\ \iff \dot{g}_{t_0, u_0}^c(t) &= F(t, g_{t_0, u_0}^c(t)) \end{aligned}$$

where $g_{t_0, u_0}^c = \phi \circ g_{t_0, u_0}^h: J_{t_0, V} \rightarrow \mathbb{R}^n$. and

$$\begin{aligned} F: I_{t_0} \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (t, x) &\mapsto -d\phi|_{\phi^{-1}(x)} \circ dR_{\phi^{-1}(x)}|_e \circ (\tilde{\gamma}_{t_0, u_0}^* \omega)_t \end{aligned} \quad (1)$$

is obviously smooth. The initial conditions become $g_{t_0, u_0}^c(t_0) = \phi(e)$. The solutions of the original equation are in bijection with the solutions of this last coordinate expression.

Now, we run the argument in reverse. F is smooth, hence uniformly Lipschitz-continuous in x and continuous in t . By the Picard-Lindelöf theorem there is an open subset of I_{t_0} containing t_0 on which

$$\dot{g}_{t_0, u_0}^c(t) = F(t, g_{t_0, u_0}^c(t)) \quad g_{t_0, u_0}^c = \phi(e)$$

has a unique solution which is C^1 . This solution is actually smooth because we can repeatedly differentiate the equation as F is smooth and use the differential equation inductively to differentiate g_{t_0, u_0}^c . Let this interval be $I_{t_0, u_0} \subseteq I_{t_0}$.

Thus there is a unique $g_{t_0, u_0}^h : I_{t_0, u_0} \rightarrow G$ which satisfies

$$dg_{t_0, u_0}^h|_t = -dR_{g_{t_0, u_0}^h(t)}|_e \circ (\tilde{\gamma}_{t_0, u_0}^* \omega)_t \quad g_{t_0, u_0}^h(t_0) = e$$

given by $g_{t_0, u_0}^h = \phi^{-1} \circ g_{t_0, u_0}^c$. Define $\gamma_{t_0, u_0}^h : I_{t_0, u_0} \rightarrow E$ as $\gamma_{t_0, u_0}^h(t) = \tilde{\gamma}_{t_0, u_0}(t) \cdot g_{t_0, u_0}^h(t)$. This is a horizontal lift, and is unique as there is only one g_{t_0, u_0}^h solving the differential equation and any lift can be written by acting on $\tilde{\gamma}_{t_0, u_0}$ by some $g_{t_0, u_0}(t)$. (Unique in the sense that given any other local lift, it has to agree with this one on the domain of intersection which is an open set around t_0 .) $\square$

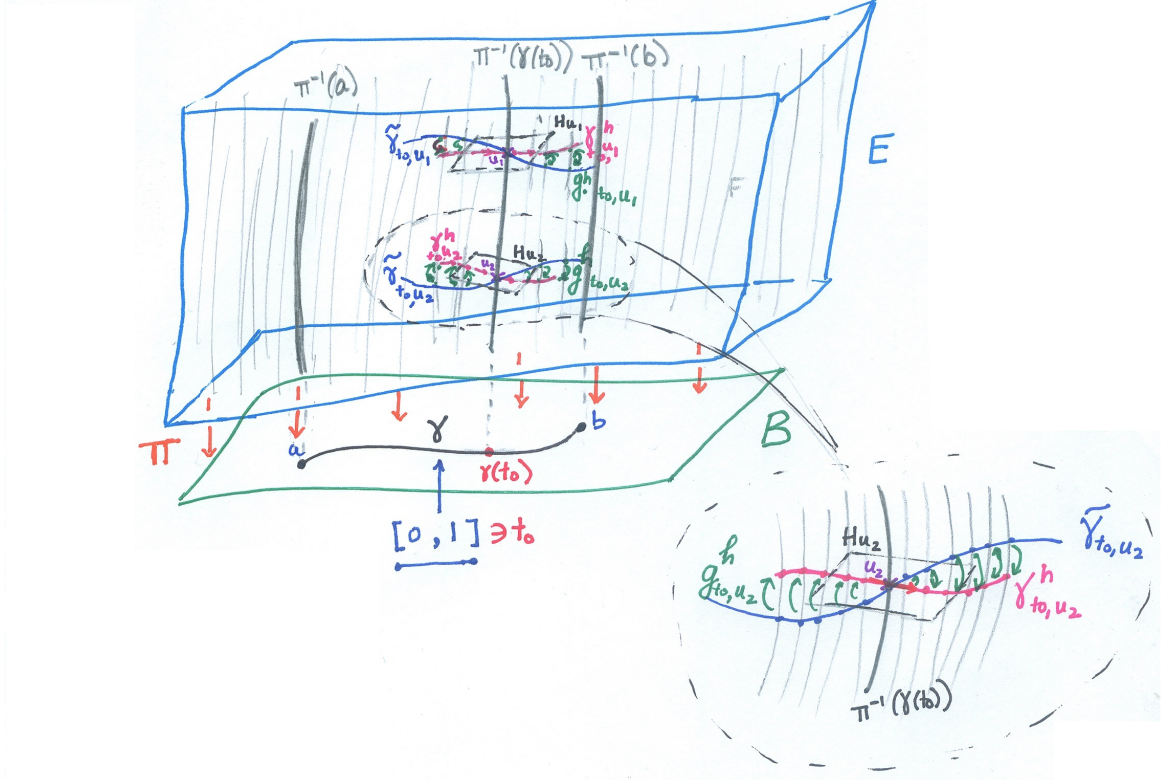


FIGURE 3. Local horizontal lifts constructed in 1.2.6

Theorem 1.2.7. — Let $(\pi: E \rightarrow B, F, a)$ be a principle G -bundle with a connection form ω . For a piecewise smooth path $\gamma: I \rightarrow B$ and a $u_0 \in \pi^{-1}(\{\gamma(0)\})$ there is a unique path $\gamma^h: I \rightarrow E$ such that

- $\gamma_{u_0}^h$ is smooth whenever γ is (Piecewise smooth)

- $\pi \circ \gamma_{u_0}^h = \gamma$ (*Lift*)
- $\gamma_{u_0}^h(0) = u_0$ (*Begins at u_0*)
- $\omega_{\gamma_{u_0}^h(t)}(d\gamma_{u_0}^h|_t(1)) = 0$ whenever defined (*Horizontal*)

Proof. — As a set,

$$E = \bigsqcup_{b \in B} \pi^{-1}(\{b\})$$

So, by the axiom of choice, there exists function $f: B \rightarrow E$ (need not be smooth) such that $f(b) \in \pi^{-1}(\{b\})$ for all $b \in B$. That is equivalent to a saying that $\pi \circ f = \text{id}_B$. Let $\gamma: I \rightarrow B$ be smooth. From 1.2.6, for every $t_0 \in I$, we have an open, connected subset of I (let it be $I_{t_0, f(\gamma(t_0))}$) and a smooth locally unique horizontal lift $\gamma_{t_0, f(\gamma(t_0))}^h: I_{t_0, f(\gamma(t_0))} \rightarrow E$ such that $\gamma_{t_0, f(\gamma(t_0))}^h(t_0) = f(\gamma(t_0))$. $\{I_{t_0, f(\gamma(t_0))}\}_{t_0 \in I}$ is an open cover of I . Since I is compact, there is a finite subcover $\{I_{t_i, f(\gamma(t_i))}\}_{i=1}^n$. Now, remember that $I_{t_i, f(\gamma(t_i))}$ are all open, connected subsets of I , so they are intervals of the form $[0, a), (a, b), (a, 1]$ for $0 < a, b < 1$. Now we define a subcover $\{J_k\}_{k=1}^m$ ($m \leq n$) as

- $J_0 = \{0\}$
- Suppose we have defined J_k for $0, \dots, k-1$. Let

$$U_{k-1} = \bigcup_{i=0}^{k-1} J_i$$

If $U_{k-1} = I$, then $m = k-1$ and we have a subcover $\{J_k\}_{k=1}^m$. Otherwise consider all the $I_{t_i, f(\gamma(t_i))}$ which have non-empty intersection with U_{k-1} and are not a subset of U_{k-1} . In the case $k = 1$, one such set exists as they are a cover and so one of them must contain 0 and an open set cannot be a subset of $\{0\}$. In the case $k > 1$, since $U_{k-1} \neq I$, the union of all the intervals which are not subsets of U_{k-1} is also open and that union along with U_{k-1} will be a disconnection of I , which is not possible. So at least one such set exists. Take the set such that $I_{t_u, f(\gamma(t_u))} \setminus U_{k-1}$ has maximum length (maximum is defined and guaranteed to exist in the cover as the cover is finite).

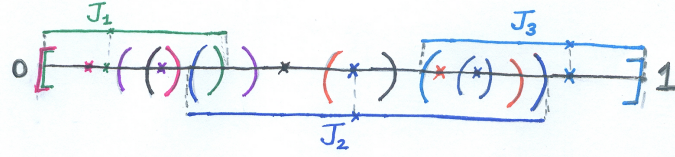


FIGURE 4. The above construction of a ‘good’ finite cover $\{J_k\}_{k=1}^m$ of I

The above inductive process will end as the cover is finite. This cover has the property that $J_k \cap J_{k'} \neq \emptyset$ iff $|k - k'| \leq 1$. Let us establish the following conventions

- Let $J_k = I_{t_{i_k}, f(\gamma(t_{i_k}))}$. Then define $t_{i_k} = \tau_k$ and $f(\gamma(t_{i_k})) = u_k$.
- Let $\gamma_{\tau_k, u_k}^h = \gamma_k^h$. So $\gamma_k^h(\tau_k) = u_k$

Let $u_0 \in \pi^{-1}(\{\gamma(0)\})$. We construct $\tilde{\gamma}_k^h: J_k \rightarrow E$ inductively as follows

- We have $\gamma_1^h: J_1 \rightarrow E$. Let $g_1 \in G$ be such that $\gamma_1^h(0) \cdot g_1 = u_0$. Define $\tilde{\gamma}_1^h: J_1 \rightarrow E$ by $\tilde{\gamma}_1^h(t) = \gamma_1^h(t) \cdot g_1$.
- Suppose we have defined $\tilde{\gamma}_k^h$. Let $\tau'_k \in J_k \cap J_{k+1}$. Let $g_{k+1} \in G$ be defined by $\tilde{\gamma}_k^h(\tau'_k) = \gamma_{k+1}^h(\tau'_k) \cdot g_{k+1}$. Then define $\tilde{\gamma}_{k+1}^h: J_{k+1} \rightarrow E$ by $\tilde{\gamma}_{k+1}^h(t) = \gamma_{k+1}^h(t) \cdot g_{k+1}$. Since $\tilde{\gamma}_k^h(\tau'_k) = \tilde{\gamma}_{k+1}^h(\tau'_k)$ and both $\tilde{\gamma}_k^h, \tilde{\gamma}_{k+1}^h$ are local lifts of γ around τ'_k they are equal on the intersection of the domains of definition (i.e., $J_k \cap J_{k+1}$) by 1.2.6.

Since all the $\tilde{\gamma}_k^h$ are equal on the intersections of their domains and they are all smooth, we can glue them into a smooth function $\gamma_{u_0}^h: I \rightarrow E$. The uniqueness follows from local uniqueness. Obviously $\gamma_{u_0}^h(0) = \tilde{\gamma}_1^h(0) = u_0$.

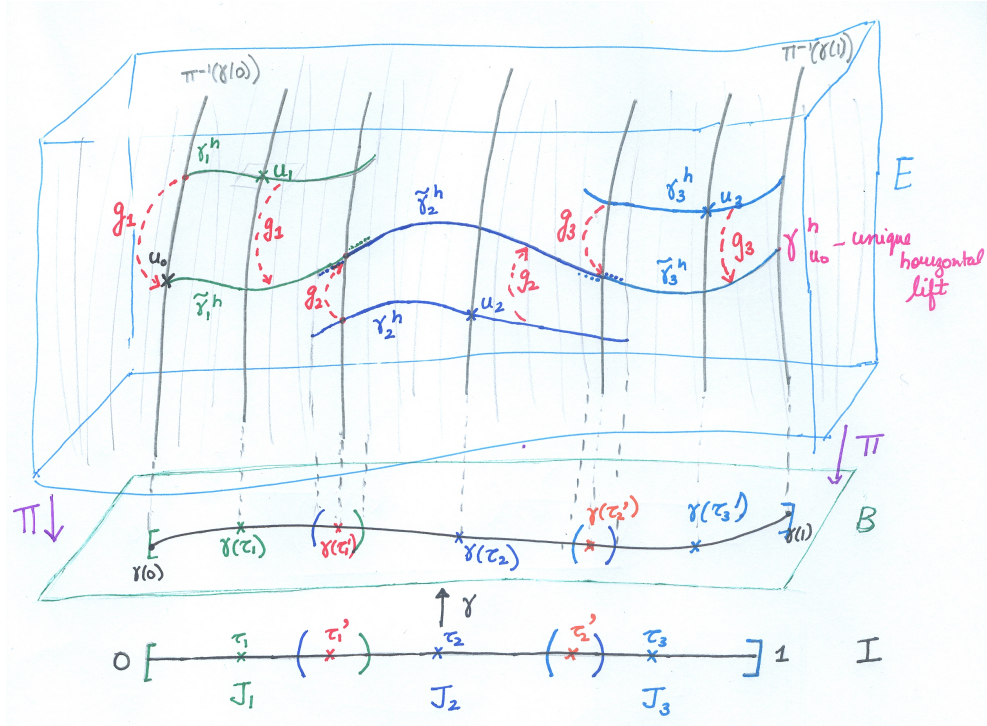


FIGURE 5. Gluing together local lifts to get a global lift

Now, let γ be only piecewise smooth, then we can break it down into finitely many smooth paths, and find their horizontal lifts. Then by a similar process of moving around in the fibres at the non-differentiable point using group elements, we can construct a unique lift which is horizontal. $\square$

Note: Suppose γ was piecewise C^1 and on its i^{th} component it was C^{k_i} , then we could have constructed (using the same sequence of proofs as above) a unique horizontal piecewise C^1 lift which was C^{k_i} above the parts of the curve that were C^{k_i} .

Corollary 1.2.8. — Let $(\pi: E \rightarrow B, F, a)$ be a principle G -bundle with a connection form ω . Let $\gamma: I \rightarrow B$ be a smooth, $\tilde{\gamma}$ be a lift and γ^h be a horizontal lift. Then the unique map $g: I \rightarrow G$ defined by $\gamma^h(t) = \tilde{\gamma}(t) \cdot g(t)$ is smooth and solves

$$dg|_t = -dR_{g(t)}|_e \circ (\tilde{\gamma}^* \omega)_t$$

Proof. — Imitate proof in first half of 1.2.6. Uniqueness follows from regularity of action and smoothness from Picard-Lindelöf. $\square$

Definition 1.2.9. — Given a piecewise C^1 curve $\gamma: I \rightarrow B$, we define a map

$$\begin{aligned} \text{tra}_\omega^\gamma: \pi^{-1}(\{\gamma(0)\}) &\rightarrow \pi^{-1}(\{\gamma(0)\}) \\ u &\mapsto \gamma_u^h(1) \end{aligned}$$

This map is called the **parallel transport along γ** .

Theorem 1.2.10 (Properties of tra). — Let $(\pi: E \rightarrow B, F, a)$ be a principle G -bundle with a connection form ω . Then,

1. If γ is piecewise C^k , then tra_ω^γ is C^k
2. If $\gamma: I \rightarrow B$ is constant (i.e., $\gamma(t) = \gamma(0)$), then $\text{tra}_\omega^\gamma = \text{id}_{\pi^{-1}(\{\gamma(0)\})}$
3. $\text{tra}_\omega^{\gamma_1 * \gamma_2} = \text{tra}_\omega^{\gamma_1} \circ \text{tra}_\omega^{\gamma_2}$
4. If γ_1 is thin homotopic (endpoint preserving) to γ_2 , then $\text{tra}_\omega^{\gamma_1} = \text{tra}_\omega^{\gamma_2}$
5. If $\rho: I \rightarrow I$ is an orientation-preserving reparameterisation, then $\text{tra}_\omega^\gamma = \text{tra}_\omega^{\gamma \circ \rho}$
6. Let $\bar{\gamma}: I \rightarrow B$ be defined by $\bar{\gamma}(t) = \gamma(1-t)$, then $\text{tra}_\omega^\gamma \circ \text{tra}_\omega^{\bar{\gamma}} = \text{id}_{\pi^{-1}(\{\gamma(0)\})}$
7. tra_ω^γ is a C^k -diffeomorphism
8. $a_g \circ \text{tra}_\omega^\gamma = \text{tra}_\omega^\gamma \circ a_g$

Note: In point 3. $\gamma_1 * \gamma_2$ is the curve obtained by going along γ_2 then γ_1 , i.e. is the concatenation of the two paths. Let $p = (\cos(\theta), \sin(\theta)) \in S^1$. Define $\rho_p: I \rightarrow S^1$ by $\rho_p(t) = (\cos(\theta + 2\pi t), \sin(\theta + 2\pi t))$.

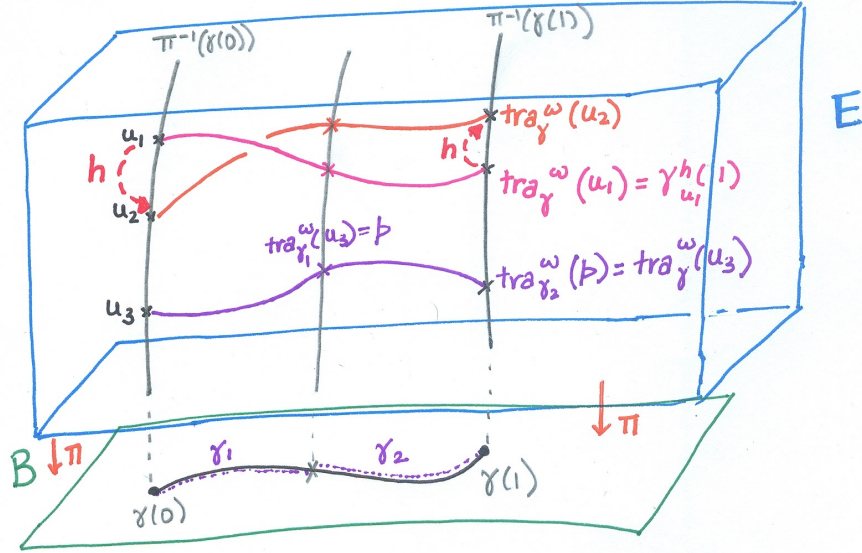


FIGURE 6. The transport map

Theorem 1.2.11 (Existence of Wilson Loops). — Let $(\pi: E \rightarrow B, F, a)$ be a principle G -bundle with a connection form ω . Let $\gamma: S^1 \rightarrow B$ be piecewise C^1 . Choose any point $p \in S^1$. Define the map

$$W: C_{pc}^1[S^1, B] \rightarrow C(G)$$

$$\gamma \mapsto h_{\omega}^{\gamma \circ \rho_p}(\pi^{-1}(\{p\}))$$

where $C(G)$ is the set of conjugacy classes of G , $C_{pc}^1[S^1, G]$ is the set of piece-wise C^1 map from $S^1 \rightarrow G$, and for a closed curve $\hat{\gamma}: I \rightarrow B$ such that $\hat{\gamma}(0) = \hat{\gamma}(1) = q$ we define,

$$h_{\omega}^{\hat{\gamma}}: \pi^{-1}(\{q\}) \rightarrow \pi^{-1}(\{q\})$$

$$\text{tra}_{\omega}^{\hat{\gamma}}(u) = u \cdot h_{\omega}^{\hat{\gamma}}(u)$$

The map W is well-defined (does not depend on the choice of p and does actually map into $C(G)$).

Proof. — For any closed curve $\hat{\gamma}$, we can transport a point u in the fibre above the base point into the same fibre. By regularity of the action, there would be unique group element linking the original point to the transported point. So $h_{\omega}^{\hat{\gamma}}$ is well defined. Let $\hat{\gamma}_p = \gamma \circ \rho_p$. Take two points $u, v \in \pi^{-1}(\{p\})$. So there

is a unique $g \in G$ such that $v = u \cdot g$ (regularity of the action).

$$\begin{aligned}\mathrm{tra}_{\omega}^{\hat{\gamma}^p}(v) &= \mathrm{tra}_{\omega}^{\hat{\gamma}^p}(u \cdot g) = (u \cdot g) \cdot h_{\omega}^{\hat{\gamma}^p}(u \cdot g) = u \cdot g h_{\omega}^{\hat{\gamma}^p}(v) \\ \mathrm{tra}_{\omega}^{\hat{\gamma}^p}(u \cdot g) &= \mathrm{tra}_{\omega}^{\hat{\gamma}^p}(u) \cdot g = \left(u \cdot h_{\omega}^{\hat{\gamma}^p}(u)\right) \cdot g = u \cdot h_{\omega}^{\hat{\gamma}^p}(u) g\end{aligned}$$

Hence, $h_{\omega}^{\hat{\gamma}^p}(v) = g^{-1} h_{\omega}^{\hat{\gamma}^p}(u) g$. So given two elements of a fibre, their images under $h_{\omega}^{\hat{\gamma}^p}$ are conjugate. Also, as the entire fibre is generated by acting by the entire group on u , we have that the image of the fibre under $h_{\omega}^{\hat{\gamma}^p}$ is the entire conjugacy class of $h_{\omega}^{\hat{\gamma}^p}(u)$. That is,

$$h_{\omega}^{\hat{\gamma}^p}(\pi^{-1}(\{p\})) = \mathrm{Cl}\left(h_{\omega}^{\hat{\gamma}^p}(u)\right)$$

for any $u \in \pi^{-1}(\{p\})$. So the W does map into $\mathbf{C}(G)$. Now we check that this class is independent of the choice of basepoint.

Let $p = (\cos(\theta_p), \sin(\theta_p))$, $q = (\cos(\theta_q), \sin(\theta_q)) \in S^1$ with $p \neq q$ and let

$$\begin{aligned}\alpha_{pq}: I &\rightarrow S^1 \\ t &\mapsto (\cos(\theta_p + (\theta_q - \theta_p)t), \sin(\theta_p + (\theta_q - \theta_p)t))\end{aligned}$$

Define $\beta_{pq} = \gamma \circ \alpha_{pq}$. Then $\hat{\gamma}_p = \beta_{qp} * \beta_{pq}$ and $\hat{\gamma}_q = \beta_{pq} * \beta_{qp}$. Up to reparameterisation, this is the unique decomposition (up to reparameterisations) of $\hat{\gamma}_p = b * a$, $\hat{\gamma}_q = a * b$ such that $a(0) = \beta(1) = p$ and $\alpha(1) = \beta(0) = q$. So,

$$\begin{aligned}\mathrm{tra}_{\omega}^{\hat{\gamma}^p} &= \mathrm{tra}_{\omega}^{\beta_{qp} * \beta_{pq}} = \mathrm{tra}_{\omega}^{\beta_{qp}} \circ \mathrm{tra}_{\omega}^{\beta_{pq}} \\ \mathrm{tra}_{\omega}^{\hat{\gamma}^q} &= \mathrm{tra}_{\omega}^{\beta_{pq} * \beta_{qp}} = \mathrm{tra}_{\omega}^{\beta_{pq}} \circ \mathrm{tra}_{\omega}^{\beta_{qp}} \\ \implies \mathrm{tra}_{\omega}^{\hat{\gamma}^p} &= \mathrm{tra}_{\omega}^{\beta_{qp}} \circ \mathrm{tra}_{\omega}^{\hat{\gamma}^q} \circ \left(\mathrm{tra}_{\omega}^{\beta_{qp}}\right)^{-1}\end{aligned}$$

Let $u \in \pi^{-1}(\{p\})$. Then,

$$\begin{aligned}\mathrm{tra}_{\omega}^{\hat{\gamma}^p}(u) &= \mathrm{tra}_{\omega}^{\beta_{qp}}\left(\mathrm{tra}_{\omega}^{\hat{\gamma}^q}\left(\left(\mathrm{tra}_{\omega}^{\beta_{qp}}\right)^{-1}(u)\right)\right) \\ \implies u \cdot h_{\omega}^{\hat{\gamma}^p}(u) &= \mathrm{tra}_{\omega}^{\beta_{qp}}\left(\left(\mathrm{tra}_{\omega}^{\beta_{qp}}\right)^{-1}(u) \cdot h_{\omega}^{\hat{\gamma}^q}\left(\left(\mathrm{tra}_{\omega}^{\beta_{qp}}\right)^{-1}(u)\right)\right) \\ &= u \cdot h_{\omega}^{\hat{\gamma}^q}\left(\left(\mathrm{tra}_{\omega}^{\beta_{qp}}\right)^{-1}(u)\right) \implies h_{\omega}^{\hat{\gamma}^p}(u) = h_{\omega}^{\hat{\gamma}^q}\left(\left(\mathrm{tra}_{\omega}^{\beta_{qp}}\right)^{-1}(u)\right)\end{aligned}$$

Thus,

$$h_{\omega}^{\hat{\gamma}^p}(\pi^{-1}(\{p\})) = \mathrm{Cl}\left(h_{\omega}^{\hat{\gamma}^p}(u)\right) = \mathrm{Cl}\left(h_{\omega}^{\hat{\gamma}^q}\left(\left(\mathrm{tra}_{\omega}^{\beta_{qp}}\right)^{-1}(u)\right)\right) = h_{\omega}^{\hat{\gamma}^q}(\pi^{-1}(\{q\}))$$

So as claimed, W is well defined as the class it maps to does not depend on the basepoint. $\square$

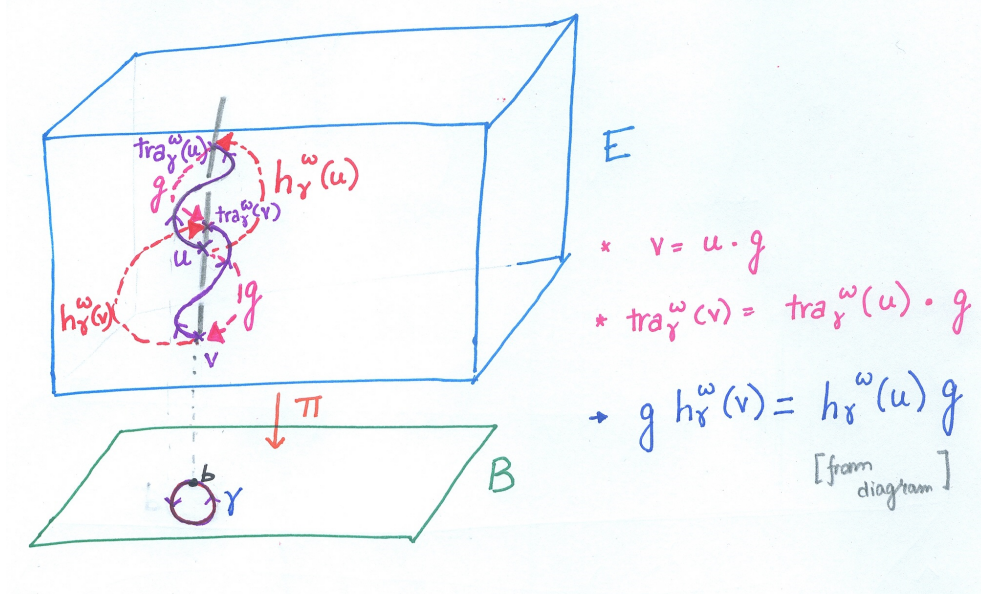


FIGURE 7. Illustrating how the image of a fibre under h_γ^ω is a conjugacy class

Definition 1.2.12. — Let (E, B, π, G) be a principal G -bundle. Choose an open cover of B , $\{U_\alpha\}$ and a local connection 1-form on each of the $\pi^{-1}(U_\alpha)$'s such that on $\pi^{-1}(U_\alpha \cap U_{\alpha'})$, the two connections are related by a gauge transformation of the bundle $\pi^{-1}(U_\alpha \cap U_{\alpha'}) \rightarrow U_\alpha \cap U_{\alpha'}$. This is, the connection one forms need to match on intersections of the opens sets only by a locally defined gauge transformation. Such a system is called a **consistent patchwork of connections** or CPoC. Let us call this $\{U_\alpha, A_\alpha\}$. Now suppose we have two such CPoCs, $\{(U_\alpha, A_\alpha)\}_{\alpha \in I}$ and $\{(V_\beta, B_\beta)\}_{\beta \in J}$ then these two are **gauge equivalent** if on every $U_\alpha \cap V_\beta$ the restricted connections $A_\alpha|_{T\pi^{-1}(U_\alpha \cap V_\beta)}$ and $B_\beta|_{T\pi^{-1}(U_\alpha \cap V_\beta)}$ are related by a local gauge transformation of the bundle $\pi^{-1}(U_\alpha \cap V_\beta) \rightarrow U_\alpha \cap V_\beta$.

A **gauge theory** is defined to be a theory whose space of (physical) fields is the *quotient* of consistent patchworks of connections by the gauge equivalence, and thus, physical quantities are gauge invariant.

1.2.1. A special case: Trivial Bundles. — Let B be a manifold and G be a Lie group. Then throughout this chapter, we will be concerned with connections on the bundle

$$F = G \quad E = B \times G \quad \pi(b, g) = b \quad (b, g) \cdot h = (b, gh)$$

Lemma 1.2.13. — *A map $f: B \times G \rightarrow B \times G$ is a gauge transformation if and only if it is of the form*

$$f(b, g) = (b, \alpha(b)g)$$

for some $\alpha: B \rightarrow G$. This α is unique.

Proof. — Let $f: B \times G \rightarrow B \times G$. By the definition of product, we can write $f(b, g) = (f_B(b, g), f_G(b, g))$ for some unique $f_B: B \times G \rightarrow B$ and $f_G: B \times G \rightarrow G$. For this to be a map of G -bundles, we need that

$$\begin{aligned} f(b, g) \cdot h &= f((b, g) \cdot h) \\ \implies (f_B(b, g), f_G(b, g)) \cdot h &= (f_B((b, g) \cdot h), f_G((b, g) \cdot h)) \\ \implies (f_B(b, g), f_G(b, g)h) &= (f_B(b, gh), f_G(b, gh)) \\ \implies f_B(b, g) &= f_B(b, gh) \wedge f_G(b, g)h = f_G(b, gh) \\ \implies f_B(b, e) &= f_B(b, h) \wedge f_G(b, e)h = f_G(b, h) \end{aligned}$$

where in the last step we set $g = e$. Let $f_1: B \rightarrow B$ and $f_2: B \rightarrow G$ be defined by $f_1(b) = f_B(b, e)$ and $\alpha(b) = f_G(b, e)$. So,

$$f(b, g) = (f_1(b), \alpha(b)g)$$

Since this is a map of G -bundles it will induce a map $\bar{f}: B \rightarrow B$. So let's see how this works.

$$\{b\} \pi^{-1} \{b\} \times G f \{f_1(b)\} \times \alpha(b) G \pi \{f_1(b)\}$$

So $\bar{f} = f_1$. For a gauge transformation, $\bar{f} = \text{id}_B$. So $f_1 = \text{id}_B$. Also, we need f to be invertible for it to be a gauge transformation. It is easy to see that the map $(b, g) \mapsto (b, \alpha(b)^{-1}g)$ is f^{-1} . Hence any gauge transformation can be written in the given form. It is easy to see that any map of this form is a gauge transformation, by just verifying the condition. Uniqueness is obvious as if there were two ways of writing $f(b, g) = (b, \alpha_1(b)g) = (b, \alpha_2(b)g)$, then $\alpha_1(b) = \alpha_2(b)$ by cancelling the g . $\square$ $\square$

Corollary 1.2.14. — *The groups G and $C^\infty(B, G)$ are isomorphic.*

Theorem 1.2.15. — *A Lie algebra valued form $\omega: T(B \times G) \rightarrow \mathfrak{g}$ is a connection form if and only if it is of the form*

$$\omega_{b,g} u_{(b,g)} = \text{Ad}_{g^{-1}} \circ A_b v_b + \theta_g w_g$$

for some form $A: TB \rightarrow \mathfrak{g}$. This A is unique. Here θ is the Maurer-Cartan form and v_b and w_g are the vectors defined in the next sentence. Let $P: T(- \times -) \xrightarrow{\sim} T- \times T-$ be the natural isomorphism. Then $P_{B,G}(u_{(b,g)}) = (v_b, w_g)$.

Proof. — Let $\omega: T(B \times G) \rightarrow \mathfrak{g}$ be a form. Define

$$\begin{aligned}\tilde{\omega}: TB \times TG &\rightarrow \mathfrak{g} \\ (v_b, w_g) &\mapsto \omega_{(b,g)} \left(P_{B,G}^{-1}(v_b, w_g) \right)\end{aligned}$$

By linearity, we can write

$$\tilde{\omega}_{(b,g)}(v_b, w_g) = \tilde{\omega}_{(b,g)}(v_b, 0_g) + \tilde{\omega}_{(b,g)}(0_b, w_g) = \tilde{A}_{(b,g)}v_b + \tilde{C}_{(b,g)}w_g$$

where $\tilde{A}_{(b,g)}v_b = \tilde{\omega}_{(b,g)}(v_b, 0_g)$ and $\tilde{C}_{(b,g)}w_g = \tilde{\omega}_{(b,g)}(0_b, w_g)$. For ω to be a connection form, we need

$$\begin{aligned}\text{Ad}_{h^{-1}} \circ \omega &= a_h^* \omega \text{ for } h \in G \\ \omega_{(b,g)} \left(X_{(b,g)}^\# \right) &= X_e \text{ for } X_e \in \mathfrak{g}\end{aligned}$$

where $a_h: B \times G \rightarrow B \times G$ defined by $a_h(b, g) = (b, gh)$. Now we need to find $X^\#$ for $X_e \in \mathfrak{g}$. Let

$$\begin{aligned}a_{(b,g)}: G &\rightarrow B \times G \\ h &\mapsto (b, gh)\end{aligned}$$

We see that $a_{(b,g)} = (k_b, L_g)$ (using the fact that a map into a product can be decomposed) where $k_b: G \rightarrow B$ is the constant at b . Thus,

$$\begin{aligned}da_{(b,g)}|_h: T_h G &\rightarrow T_{(b,gh)}(B \times G) \\ w_h &\mapsto P_{B,G}^{-1}(0_b, dL_g|_h w_h)\end{aligned}$$

This is easy to see from the fact that T preserves products. So, if $X_e \in \mathfrak{g}$, that is, $X_e \in T_e G$, then

$$X_{(b,g)}^\# = da_{(b,g)}|_e X_e = P_{B,G}^{-1}(0_b, dL_g|_e X_e)$$

Now,

$$\tilde{\omega}_{(b,g)} \left(P_{B,G} \left(X_{(b,g)}^\# \right) \right) = \tilde{A}_{(b,g)}(0_b) + \tilde{C}_{(b,g)}(dL_g|_e X_e) = \tilde{C}_{(b,g)}(dL_g|_e X_e)$$

So the second condition for ω to be a connection form becomes

$$\begin{aligned}\tilde{C}_{(b,g)}(dL_g|_e X_e) &= X_e \text{ for } X_e \in \mathfrak{g} \\ \implies \tilde{C}_{(b,g)} &= (dL_g|_e)^{-1} = dL_{g^{-1}}|_g = \theta_g\end{aligned}$$

where the second last equality follows from the chain rule and θ is the Maurer-Cartan form. Now we turn to the other condition on ω . As we had already defined,

$$\begin{aligned}a_h: B \times G &\rightarrow B \times G \\ (b, g) &\mapsto (b, gh)\end{aligned}$$

We see that $a_h = (\text{proj}_B, R_h \circ \text{proj}_G)$ (by decomposing the product) and then using the fact that T preserves products and using linearity, we get

$$\begin{aligned} da_h|_{(b,g)}: T_{(b,g)}(B \times G) &\rightarrow T_{(b,gh)}(B \times G) \\ P_{B,G}^{-1}(v_b, w_g) &\mapsto P_{B,G}^{-1}(v_b, dR_h|_g w_g) \end{aligned}$$

So,

$$\begin{aligned} (a_h^* \omega)_{(b,g)}(P_{B,G}^{-1}(v_b, w_g)) &= (\omega_{(b,gh)} \circ da_h|_{(b,g)})(P_{B,G}^{-1}(v_b, w_g)) \\ &= \tilde{\omega}_{(b,gh)}(v_b, dR_h|_g w_g) = \tilde{A}_{(b,gh)} v_b + \tilde{C}_{(b,gh)}(dR_h|_g w_g) \end{aligned}$$

The other half of the first condition for ω to be a connection form is

$$(\text{Ad}_{h^{-1}} \circ \omega)_{(b,g)}(P_{B,G}^{-1}(v_b, w_g)) = \text{Ad}_{h^{-1}}(\tilde{A}_{(b,g)} v_b + \tilde{C}_{(b,g)} w_g)$$

Since all the maps are linear, the only way for the above two halves to be equal is if

$$\begin{aligned} \tilde{A}_{(b,gh)} v_b &= \text{Ad}_{h^{-1}}(\tilde{A}_{(b,g)} v_b) \\ \tilde{C}_{(b,gh)}(dR_h|_g w_g) &= \text{Ad}_{h^{-1}}(\tilde{C}_{(b,g)} w_g) \end{aligned}$$

Or, we have the following equalities

$$\begin{aligned} \tilde{A}_{(b,gh)} &= \text{Ad}_{h^{-1}} \circ \tilde{A}_{(b,g)} \\ \tilde{C}_{(b,gh)} \circ dR_h|_g &= \text{Ad}_{h^{-1}} \circ \tilde{C}_{(b,g)} \end{aligned}$$

We already have $\tilde{C}_{(b,g)} = dL_{g^{-1}}|_g$. We now check that it satisfies this condition too. So,

$$\begin{aligned} \tilde{C}_{(b,gh)} \circ dR_h|_g &= dL_{(gh)^{-1}}|_{gh} \circ dR_h|_g = d(L_{(gh)^{-1}} \circ R_h)|_g \\ &= d(c_h \circ L_{g^{-1}})|_g = dc_h|_e \circ dL_{g^{-1}}|_g = \text{Ad}_{h^{-1}} \circ \tilde{C}_{(b,g)} \end{aligned}$$

where c_g is conjugation by g . Thus the second property is verified. Now, let's understand the condition on $\tilde{A}$ as $\tilde{C}$ has been fully determined.

$$\tilde{A}_{(b,gh)} = \text{Ad}_{h^{-1}} \circ \tilde{A}_{(b,g)} \implies \tilde{A}_{(b,h)} = \text{Ad}_{h^{-1}} \circ \tilde{A}_{(b,e)}$$

We get the implication by setting $g = e$. It is easy to check that given any choice of form $A: TB \rightarrow \mathfrak{g}$, we can define $\tilde{A}_{(b,g)} = \text{Ad}_{g^{-1}} \circ A_b$. It is easy to see that, this satisfies the condition on $\tilde{A}$ and this the only possibility because of the above implication. So any connection form can be written as

$$\omega_{(b,g)}(P_{B,G}^{-1}(v_b, w_g)) = \text{Ad}_{g^{-1}}(A_b v_b) + \theta_g w_g$$

for some form $A: TB \rightarrow \mathfrak{g}$ and any form of the structure given above is a connection form. Uniqueness follows as is there are two A s, giving the same omega, the $\text{Ad}_{g^{-1}}$ will cancel and the A s will be equal. $\square$ $\square$

Theorem 1.2.16. — *For a connection form ω and a gauge transformation f such that*

$$\begin{aligned}\omega_{(b,g)} \left(P_{B,G}^{-1}(v_b, w_g) \right) &= \text{Ad}_{g^{-1}}(A_b v_b) + \theta_g w_g \\ f(b, g) &= (b, \alpha(b)g)\end{aligned}$$

the pullback connection form is given by

$$\begin{aligned}(f^* \omega)_{(b,g)} \left(P_{B,G}^{-1}(v_b, w_g) \right) &= \text{Ad}_{g^{-1}}(A'_b v_b) + \theta_g w_g \\ \text{where } A'_b &= \text{Ad}_{\alpha(b)^{-1}} \circ A_b + (\alpha^* \theta)_b\end{aligned}$$

Proof. — We first need to compute $df_\alpha|_{(b,g)}$. We have $f_\alpha = (\text{proj}_B, \mu \circ (\alpha \times \text{id}_G))$ where $\mu: G \times G \rightarrow G$ is the group multiplication. Since it is map into a product, it is enough to differentiate its components. Using the chain rule to differentiate the second component, we get

$$df_\alpha|_{(b,g)} \left(P_{B,G}^{-1}(v_b, w_g) \right) = P_{B,G}^{-1}(v_b, d\mu|_{(\alpha(b),g)}(d\alpha|_b v_b, w_g))$$

Now let's compute a pullback of a connection form. So,

$$\begin{aligned}(f_\alpha^* \omega)_{(b,g)} \left(P_{B,G}^{-1}(v_b, w_g) \right) &= (\omega_{(b,\alpha(b)g)} \circ df_\alpha|_{(b,g)}) \left(P_{B,G}^{-1}(v_b, w_g) \right) \\ &= \tilde{\omega}_{(b,\alpha(b)g)} \left(v_b, d\mu|_{(\alpha(b),g)} \left(P_{G,G}^{-1}(d\alpha|_b v_b, w_g) \right) \right) \\ &= \tilde{A}_{(b,\alpha(b)g)} v_b + \tilde{C}_{(b,\alpha(b)g)} \left(d\mu|_{(\alpha(b),g)} \left(P_{B,G}^{-1}(d\alpha|_b v_b, w_g) \right) \right)\end{aligned}$$

Let $f_\alpha^* \omega = \omega'$ and

$$\tilde{\omega}'_{(b,g)}(v_b, w_g) = \tilde{A}'_{(b,g)} v_b + \tilde{C}'_{(b,g)} w_g$$

Then using linearity, we get that the following relations

$$\begin{aligned}\tilde{A}'_{(b,g)} v_b &= \tilde{A}_{(b,\alpha(b)g)} v_b + \tilde{C}_{(b,\alpha(b)g)} \left(d\mu|_{(\alpha(b),g)} \left(P_{G,G}^{-1}(d\alpha|_b v_b, 0_g) \right) \right) \\ \tilde{C}'_{(b,g)} w_g &= \tilde{C}_{(b,\alpha(b)g)} \left(d\mu|_{(\alpha(b),g)} \left(P_{G,G}^{-1}(0_{\alpha(b)}, w_g) \right) \right)\end{aligned}$$

Define a map $\delta_h: G \rightarrow G \times G$ by $\delta_h(g) = (h, g)$. Obviously, $d\delta_h|_g w_g = P_{G,G}^{-1}(0_h, w_g)$. So we can rewrite the second transformation equation as

$$\begin{aligned}\tilde{C}'_{(b,g)} &= \tilde{C}_{(b,\alpha(b)g)} \circ d\mu|_{(\alpha(b),g)} \circ d\delta_{\alpha(b)}|_g \\ &= dL_{(\alpha(b)g)^{-1}}|_{\alpha(b)g} \circ d\mu|_{(\alpha(b),g)} \circ d\delta_{\alpha(b)}|_g \\ &= d \left(L_{(\alpha(b)g)^{-1}} \circ \mu \circ \delta_{\alpha(b)} \right) |_g = dL_{g^{-1}}|_g = \tilde{C}_{(b,g)}\end{aligned}$$

So we see that $\tilde{C}$ doesn't change under a gauge transformation and so still satisfies the properties required for it to add with $\tilde{A}$ to get a connection. Now, we need to see how $\tilde{A}$ transforms and if it also changes such that $\tilde{A} + \tilde{C}$

remains a connection. Define $\eta_h: B \rightarrow G \times G$ by $\eta_h(b) = (\alpha(b), h)$. Obviously, $d\eta_h|_b v_b = P_{G,G}^{-1}(d\alpha|_b v_b, 0_h)$. So we can rewrite the transformation of $\tilde{A}_{(b,g)}$ as

$$\begin{aligned}
\tilde{A}'_{(b,g)} &= \tilde{A}_{(b,\alpha(b)g)} + \tilde{C}_{(b,\alpha(b)g)} \circ d\mu|_{(\alpha(b),g)} \circ d\eta_g|_b \\
&= \tilde{A}_{(b,\alpha(b)g)} + dL_{(\alpha(b)g)^{-1}}|_{\alpha(b)g} \circ d\mu|_{(\alpha(b),g)} \circ d\eta_g|_b \\
&= \tilde{A}_{(b,\alpha(b)g)} + d\left(L_{(\alpha(b)g)^{-1}} \circ \mu \circ \eta_g\right)|_b \\
&= \tilde{A}_{(b,\alpha(b)g)} + d\left(c_{g^{-1}} \circ L_{\alpha(b)^{-1}} \circ \alpha\right)|_b \\
&= \tilde{A}_{(b,\alpha(b)g)} + \text{Ad}_{g^{-1}} \circ d\left(L_{\alpha(b)^{-1}} \circ \alpha\right)|_b \\
&= \text{Ad}_{(\alpha(b)g)^{-1}} \circ A_b + \text{Ad}_{g^{-1}} \circ d\left(L_{\alpha(b)^{-1}} \circ \alpha\right)|_b \\
&= \text{Ad}_{g^{-1}} \circ \left(\text{Ad}_{\alpha(b)^{-1}} \circ A_b + d\left(L_{\alpha(b)^{-1}} \circ \alpha\right)|_b\right)
\end{aligned}$$

Thus $\tilde{A}'_{(b,g)} = \text{Ad}_{g^{-1}} \circ A'_b$ for

$$A'_b = \text{Ad}_{\alpha(b)^{-1}} \circ A_b + d\left(L_{\alpha(b)^{-1}} \circ \alpha\right)|_b = \text{Ad}_{\alpha(b)^{-1}} \circ A_b + (\alpha^* \theta)_b$$

This proves the theorem. $\square$

$\square$

CHAPTER 2

KNOT THEORY

Definition 2.0.1. — A **knot** is an *embedding* of the circle S^1 into the Euclidean space $\mathbb{R}^3$, or equivalently into the 3-sphere S^3 (one can also consider knots in general 3-manifolds).

A **link** is a *smooth-embedding* of $S^1 \sqcup S^1 \dots \sqcup S^1 \rightarrow \mathbb{R}^3$ where $S^1 \sqcup S^1 \dots \sqcup S^1$ denotes a disjoint union of several circles.

Two knots are said to be isotopic (of the same isotopy type) if one can be transformed into another by means of a *smooth isotopy*. A **knot invariant** is map from isotopy equivalence classes of knots to any kind of mathematical structure. These are helpful because it is often much easier to check that the structures one maps to (numbers, groups, polynomials etc.) are different than it is to check that knots are different.

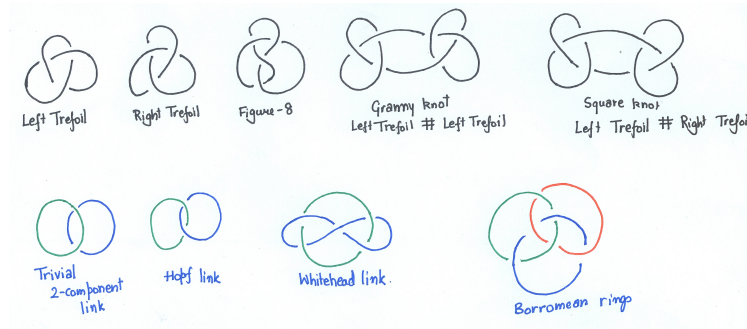


FIGURE 1. Diagrammatic representations of Knots and Links

Definition 2.0.2. — A **knot diagram** is a plane curve whose only singularities are transversal double points (crossings), together with the choice of one branch of the curve at each crossing. The chosen branch is called an

overcrossing; the other branch is referred to as an **undercrossing**. A knot diagram is thought of as a projection of a knot along some ‘vertical’ direction; overcrossings and undercrossings indicate which branch is ‘higher’ and which is ‘lower’. To indicate the orientation, an arrow is added to the knot diagram.

Theorem 2.0.3 (Reidemeister). — *Two unoriented knots K_1 and K_2 are isotopic iff we can get from a diagram of the first to a diagram of the second by finite sequence of the following moves*

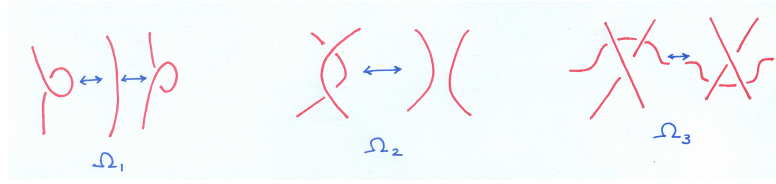


FIGURE 2. The Reidemeister Moves

Proof. — Omitted. □

Definition 2.0.4. — Kauffman Bracket: For a link diagram D , we define $\langle D \rangle$ inductively as

- $\langle \rangle = 1$
- $\langle O \sqcup D \rangle = (q + q^{-1}) \langle D \rangle$
- $\langle \text{crossing} \rangle = \langle 0\text{-smoothing} \rangle - q \langle 1\text{-smoothing} \rangle$

The third point is the following:

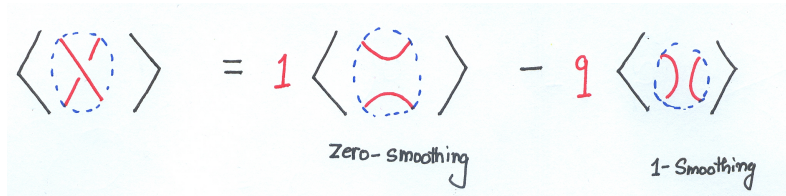


FIGURE 3. The Reidemeister Moves

Lemma 2.0.5 (Closed form for $\langle D \rangle$). — *For any link diagram D*

$$\langle D \rangle = \sum_{k=0}^n (-q)^k \sum_{S \in [k]} (q + q^{-1})^{C_S}$$

where n is the number of crossings, $[k]$ are those diagrams obtained from D after k 1-smoothings and are completely resolved (disjoint unions of circles) and C_S is the number of components in the resolved diagram S .

Proof. — Self-explanatory from induction. $\square$

Theorem 2.0.6 (Jones). — For an oriented link L , define the polynomial

$$J_L(q) = (-q^2)^{-n_-} q^{n_+} \langle D_L \rangle$$

where D_L is any oriented link diagram for L , $n_{\pm}$ is the number of positive/negative crossings in D_L and positive and negative crossings are defined as

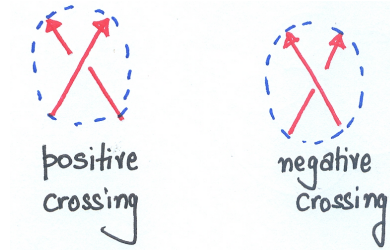


FIGURE 4. The positive and negative crossings for oriented strands

$J_L(q)$ as defined above is an invariant for knots and oriented links. To motivate the above definition of the Jones polynomial we first find the changes in the Kauffman bracket under the Reidemeister moves Ω_1 , Ω_2 , Ω_3 . It is shown further that the Kauffman Bracket remains invariant under the third move Ω_3 and changes under Ω_1 and Ω_2 . This change depends on the type of crossing resolved, i.e. positive or negative. So, in order to construct the Jones polynomial as an invariant under all three moves, we modify the Kauffman Bracket by introducing coefficients in form of powers of q that bring the positive and negative crossings in Ω_1 to the same footing as the straight stand. It is observed that the second move Ω_2 also changes by the factor of a power of q which is the same as the factors coming from Ω_1 for 1 positive crossing and 1 negative crossing in the diagram/part of the diagram being resolved.

Kauffman Bracket under R-moves

Ω_1 :

$$\begin{aligned}
 \langle \text{diag}_1 \rangle &= \langle \text{diag}_2 \rangle - q \langle \text{diag}_3 \rangle \\
 &= \langle \text{diag}_4 \rangle - q(q+q^{-1}) \langle \text{diag}_5 \rangle \\
 &= \langle \text{diag}_4 \rangle - q^2 \langle \text{diag}_5 \rangle - \langle \text{diag}_5 \rangle \\
 &= -q^2 \langle \text{diag}_5 \rangle
 \end{aligned}$$

Put orientation

$$\langle \text{diag}_1 \rangle = -q^2 \langle \text{diag}_5 \rangle$$

1 neg. crossing

$$\begin{aligned}
 \langle \text{diag}_1 \rangle &= \langle \text{diag}_3 \rangle - q \langle \text{diag}_2 \rangle \\
 &= (q+q^{-1}) \langle \text{diag}_5 \rangle - q \langle \text{diag}_4 \rangle \\
 &= q \langle \text{diag}_5 \rangle + q^{-1} \langle \text{diag}_5 \rangle - q \langle \text{diag}_4 \rangle \\
 &= q^{-1} \langle \text{diag}_5 \rangle
 \end{aligned}$$

Put orientation

$$\langle \text{diag}_1 \rangle = q^{-1} \langle \text{diag}_5 \rangle$$

1 - positive crossing

Therefore

$$\langle \text{diag}_4 \rangle = (q^{-1})^{-1} \langle \text{diag}_2 \rangle = (-q^2)^{-1} \langle \text{diag}_1 \rangle$$

pos. crossing neg. crossing

FIGURE 5. The Reidemeister Move Ω_1 on the Kauffman bracket

Ω_2 :

$$\begin{aligned}
 \langle \text{crossing} \rangle &= \langle \text{positive crossing} \rangle - q \langle \text{negative crossing} \rangle \\
 &= \langle \text{positive crossing} \rangle - q \langle \text{positive crossing} \rangle \\
 &\quad - q \langle \text{negative crossing} \rangle + q^2 \langle \text{negative crossing} \rangle \\
 &= (1 - q(q + q^{-1})) \langle \text{positive crossing} \rangle \\
 &\quad - q \langle \text{negative crossing} \rangle + q^2 \langle \text{negative crossing} \rangle \\
 &= (1 - q^2 - 1 + q^2) \langle \text{positive crossing} \rangle - q \langle \text{negative crossing} \rangle \\
 &= -q \langle \text{negative crossing} \rangle
 \end{aligned}$$

Also,

$$\langle \text{crossing} \rangle = \langle \text{positive crossing} \rangle = -q \langle \text{negative crossing} \rangle$$

On putting orientation

$$\langle \text{oriented crossing} \rangle = (-q)^{-1} \langle \text{oriented crossing} \rangle = (-q)^{-1} \langle \text{oriented crossing} \rangle$$

$+ \begin{matrix} \text{1 pos. crossing} \\ \text{1 neg. crossing} \end{matrix}$
 $+ \begin{matrix} \text{1 pos. crossing} \\ \text{1 neg. crossing} \end{matrix}$

$$(-q)^{-1} = (q^{-1})^{-1} (-q^2)^{-1}$$

$\Omega_3:$

$$\langle \text{Diagram 1} \rangle = \langle \text{Diagram 2} \rangle - q \langle \text{Diagram 3} \rangle$$

↓ using Ω_2

$$= -q \langle \text{Diagram 4} \rangle - q \langle \text{Diagram 3} \rangle$$

Also,

$$\langle \text{Diagram 1} \rangle = \langle \text{Diagram 2} \rangle - q \langle \text{Diagram 3} \rangle$$

↓ using Ω_2

$$= -q \langle \text{Diagram 4} \rangle - q \langle \text{Diagram 3} \rangle$$

Hence

$$\langle \text{Diagram 1} \rangle = \langle \text{Diagram 3} \rangle$$

FIGURE 6. The Reidemeister Move Ω_3 on the Kauffman bracket

CHAPTER 3

CHERN SIMONS GAUGE THEORY

In Physics, the starting point for defining a quantum field theory is the choice of a base manifold B of dimension n . The next is the choice of objects over B . In QFT one aims to integrate over the space parametrizing these objects (some kind of a moduli space). These objects are called fields and the operation of integration over the space of fields is also called the path-integral. In integrating over the field space we have to choose a measure on it. The measure is also usually weighted by $\exp(iS)$, where S is a functional on the space of fields in question and is called the action.

In the present case, we shall take a principal G -bundle over B to be our base space. Quantum field theories associated with connections and sections of associated vector bundles (by choosing representations of the group G) are called quantum gauge theories. Here, the path-integral is over the space of gauge equivalence classes of connections (the space of connections modulo gauge transformations). In physics terminology the choice of the connection is called picking a gauge field and a choice of a point from each orbit of the gauge transformations is called a gauge fixing condition.

The basic strategy to get a topological invariant out of a QFT(which goes hand-in-hand with the basic crux of doing the path integral) is that if we endow the space with extra geometric structures (such as a connection), then compute a number from this manifold-with-connection (using the action functional, usually itself an integral) and finally sum over all geometric structures (in this case, equivalence classes of connections) on B to take away the dependence on the added structure.

For example, suppose we have a principal G -bundle $E \xrightarrow{\pi} B$ then we can give it additional structure in the form of a connection. If B is even dimensional,

we can associate to the connection, the integral of the Chern-Weil form. If it odd dimensional, we can associate to the connection, the integral of the Chern-Simons form. Now, if we want to only associate invariants to B , we can choose $E = B \times G$, i.e. the trivial bundle. Let's look at the case for 3-manifolds. We give it a trivial G -bundle structure and take the action to be the Chern-Simons form

$$S_{\text{CS}}[A] = \frac{1}{4\pi} \int_B dA \wedge A + \frac{2}{3} A \wedge A \wedge A$$

Then the path integral will be give by

$$Z_B(k) = \int_{\mathcal{C}/\mathcal{G}} \mathcal{D}A e^{ikS_{\text{CS}}[A]} = \frac{1}{\text{Vol}(\mathcal{G})} \int_{\mathcal{C}} \mathcal{D}A e^{ikS_{\text{CS}}[A]}$$

where, $\text{Vol}(\mathcal{G})$ is the volume of the infinite-dimensional group of gauge transformations, $\mathcal{C}$ is the space of connection forms, $\mathcal{D}A$ is the measure in the space of all connections, B is the base manifold and $S_{\text{CS}}[A]$ is the action. This is well defined as long as $k \in \mathbb{Z}$ as $S_{\text{CS}}[A]$ changes by an integer under gauge transformations.

To understand the Jones polynomial quantum field theoretically, we will compute the expectation value of the Wilson loop operator (holonomy of the connection for the loop which will be a knot in the base space B). The final idea is given in the following equation:

$$J_K(k) = \int_{\mathcal{C}/\mathcal{G}} \mathcal{D}A \text{Tr}_\rho \left(\text{P exp} \left(\oint_K A \right) \right) \exp \left(\frac{ik}{4\pi} \int_{\mathbb{R}^3} \text{Tr}_\rho \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right)$$

where ρ is the fundamental representation of $\mathfrak{g}$ to get matrices from the corresponding lie-algebra elements.

$$U_{\gamma}(x_1, x_2) = \mathcal{P} \exp \left(- \int_{\gamma|[\lambda_1, \lambda_2]} A \right) \quad \begin{aligned} \gamma(\lambda_1) &= x_1 \\ \gamma(\lambda_2) &= x_2 \end{aligned}$$

$$1. \quad \frac{\delta U_{\gamma}(x_1, x_2)}{\delta x^{\alpha}(\lambda)} = - U_{\gamma}(\gamma(\lambda), x_2) F_{\alpha\beta}(x^{\sigma}(\lambda)) \dot{x}^{\beta}(\lambda) U_{\gamma}(x_1, \gamma(\lambda))$$

if $\lambda \in [\lambda_1, \lambda_2]$
where $F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} + [A_{\alpha}, A_{\beta}]$

$$= - \dot{x}^{\beta}(\lambda) F_{\alpha\beta}^a(x^{\sigma}(\lambda)) U_{\gamma}(\gamma(\lambda), x_2) T_a U_{\gamma}(x_1, \gamma(\lambda))$$

Pf. $U_{\gamma}(x_1, x_2) = \lim_{N \rightarrow \infty} \prod_{n=N-1}^0 \left(1 - \varepsilon A_{\gamma(\lambda_1 + \varepsilon n)}(\dot{\gamma}(\lambda_1 + \varepsilon n)) \right)$

where $\varepsilon = \frac{\lambda_2 - \lambda_1}{N}$

In coordinate

$$U_{\gamma}(x_1, x_2) = \lim_{N \rightarrow \infty} \prod_{n=N-1}^0 \left(1 - \varepsilon A_{\mu}^a(x^{\sigma}(\lambda_1 + \varepsilon n)) T_a \dot{x}^{\mu}(\lambda_1 + \varepsilon n) \right)$$

Therefore variation of $U_{\gamma}(x_1, x_2)$ given.

$$\frac{\delta U_{\gamma}(x_1, x_2)}{\delta x^{\alpha}(\lambda)} = \lim_{N \rightarrow \infty} \sum_{m=0}^{N-1} \prod_{n=N-1}^{m+1} \left(\begin{aligned} & - \varepsilon T_a \partial_{\alpha} A_{\mu}^a(x^{\sigma}(\lambda_1 + \varepsilon n)) \delta_{\alpha}^{\mu} \delta(\lambda_1 + \varepsilon n - \lambda) \\ & \dot{x}^{\mu}(\lambda_1 + \varepsilon n) \\ & - \varepsilon A_{\mu}^a(x^{\sigma}(\lambda_1 + \varepsilon n)) T_a \delta_{\alpha}^{\mu} \delta'(\lambda_1 + \varepsilon n - \lambda) \end{aligned} \right)$$

$$\prod_{n=m-1}^0 (\quad) .$$

$$= - \int_{\lambda_1}^{\lambda_2} U_{\gamma}(\gamma(\lambda'), x_2) T_a U_{\gamma}(x_1, \gamma(\lambda')) d\lambda' \left[\begin{aligned} & \partial_{\alpha} A_{\mu}^a(x^{\sigma}(\lambda')) \dot{x}^{\mu}(\lambda') \delta(\lambda' - \lambda) \\ & + A_{\alpha}^a(x^{\sigma}(\lambda')) \delta'(\lambda' - \lambda) \end{aligned} \right]$$

2. Expectation

$$\left\langle F_{\mu\nu}^a(x) \right\rangle = \frac{4\pi i}{k} \varepsilon_{\mu\nu\rho} \delta^{ab} \left\langle \frac{\delta \mathcal{O}}{\delta A_\rho^b(x)} \right\rangle$$

Pf: LHS =

$$\varepsilon_{\mu\nu\rho} \frac{\delta S_{CS}[A]}{\delta A_\rho^b(x)} = \frac{k}{4\pi} F_{\mu\nu}^a \delta_{ab}$$

$$S_{CS}[A] = \frac{k}{4\pi} \text{Tr} \left[\int A \wedge F - \frac{1}{3} A \wedge A \wedge A \right]$$

$$\rightarrow Z_0^{-1} \int \mathcal{D}A e^{iS_{CS}} F_{\mu\nu}^a(x) \mathcal{O}$$

$$Z_0 = \int \mathcal{D}A e^{iS[A]}$$

$$= Z_0^{-1} \int \mathcal{D}A e^{iS_{CS}} \frac{\delta S_{CS}}{\delta A_\rho^b(x)} \frac{4\pi}{k} \varepsilon_{\mu\nu\rho} \delta^{ab} \mathcal{O}$$

$$= Z_0^{-1} \int \mathcal{D}A \frac{\delta e^{iS_{CS}}}{\delta A_\rho^b(x)} \frac{4\pi}{ik} \varepsilon_{\mu\nu\rho} \delta^{ab} \mathcal{O}$$

$$= \cancel{0} + \frac{4\pi i}{k} \varepsilon_{\mu\nu\rho} \delta^{ab} Z_0^{-1} \int \mathcal{D}A e^{iS_{CS}} \frac{\delta \mathcal{O}}{\delta A_\rho^b(x)}$$

$$= \frac{4\pi i}{k} \varepsilon_{\mu\nu\rho} \delta^{ab} \left\langle \frac{\delta \mathcal{O}}{\delta A_\rho^b(x)} \right\rangle$$

(boundary of $S^3 = \emptyset$)
therefore fields on the boundary
don't exist.

3. Using (1) and (2)

$$\left\langle \frac{\delta U_\gamma(x_1, x_2)}{\delta x^\alpha(\lambda)} \right\rangle = -\dot{x}^\beta(\lambda) \left\langle F_{\alpha\beta}^a(x^\sigma(\lambda)) U_\gamma(\gamma(\lambda), x_2) T_a U_\gamma(x_1, \gamma(\lambda)) \right\rangle$$

independent of A

→ from (1)

$$= -\dot{x}^\beta(\lambda) \frac{4\pi i}{k} \delta^{ab} \epsilon_{\alpha\beta\rho} \left\langle \frac{\delta(U_\gamma(\gamma(\lambda), x_2) T_a U_\gamma(x_1, \gamma(\lambda)))}{\delta A_\rho^b(x^\sigma(\lambda))} \right\rangle$$

4. $\frac{\delta(U_\gamma(\gamma(\lambda), x_2))}{\delta A_\rho^b(x^\sigma(\lambda))} = -\dot{x}^\rho(\lambda) U_\gamma(\gamma(\lambda), x_2) T_b$

and $\frac{\delta(U_\gamma(x_1, \gamma(\lambda)))}{\delta A_\rho^b(x^\sigma(\lambda))} = 0$

Pf. $U_\gamma(\gamma(\lambda), x_2) = \lim_{N \rightarrow \infty} \prod_{n=N-1}^0 (1 - \epsilon A_\mu^a(x^\sigma(\lambda + \epsilon n)) T_a \dot{x}^\mu(\lambda + \epsilon n))$

where $\epsilon = \frac{\lambda_2 - \lambda}{N}$

$$\frac{\delta(1 - \epsilon A_\mu^a(x^\sigma(\lambda + \epsilon n)) T_a \dot{x}^\mu(\lambda + \epsilon n))}{\delta A_\rho^b(x^\sigma(\lambda))}$$

$$= -\epsilon \delta_\mu^\rho \delta_b^a T_a \dot{x}^\mu(\lambda + \epsilon n) \delta(\epsilon n)$$

$$= -\epsilon \dot{x}^\rho(\lambda + \epsilon n) T_b \delta(\epsilon n)$$

Always = 0
unless $n=0$

when $n=0$: $= -\dot{x}^\rho(\lambda + \epsilon n) T_b$

$\left\{ \begin{array}{l} \epsilon \delta(0) = 1 \end{array} \right.$

$$\frac{\delta U_Y(\gamma(\lambda), x_2)}{\delta A_p^b(x^\sigma(\lambda))}$$

$$= -\dot{x}^p(\lambda) U_Y(\gamma(\lambda), x_2) T_b$$

Similarly

$$\frac{\delta U_Y(x_1, \gamma(\lambda))}{\delta A_p^b(x^\sigma(\lambda))} = 0$$

$$\delta(\underbrace{\lambda_1 + \varepsilon n}_{\neq 0} - \lambda)$$

$$\text{where } \varepsilon = \frac{\lambda - \lambda_1}{N}$$

5. using (3) & (4)

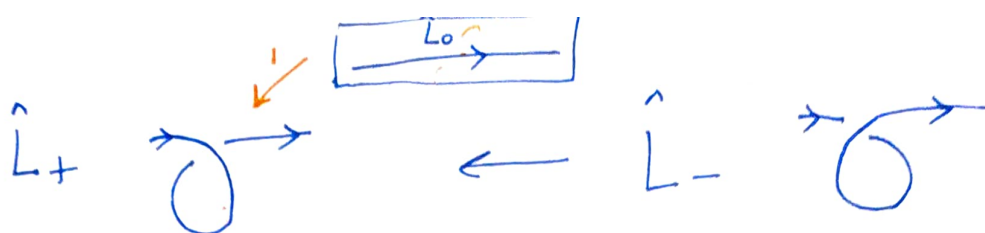
$$\left\langle \frac{\delta U_Y(x_1, x_2)}{\delta x^\alpha(\lambda)} \right\rangle = \frac{4\pi i}{k} \varepsilon_{\alpha\beta\rho} \dot{x}^\beta(\lambda) \dot{x}^\rho(\lambda) \left\langle U_Y(\gamma(\lambda), x_2) \frac{T_b T_a}{U_Y(x_1, \gamma(\lambda))} \delta^{ab} \right\rangle$$

$$\delta^{ab} T_a T_b = C_2(R) \mathbb{I} \quad \text{Casimir Operator.}$$

↳ commutes with everything (using Jacobi Identity)

By Schur's lemma it has to be a multiple of identity matrix.

$$\therefore \left\langle \frac{\delta U_Y(x_1, x_2)}{\delta x^\alpha(\lambda)} \right\rangle = \frac{4\pi i}{k} C_2(R) \varepsilon_{\alpha\beta\rho} \dot{x}^\beta(\lambda) \dot{x}^\rho(\lambda) \langle U_Y(x_1, x_2) \rangle$$



$$v = -1$$

6.

$$\langle W(\hat{L}_+) \rangle - \langle W(\hat{L}_0) \rangle = -\frac{1}{2} \times \frac{4\pi i}{k} c_2(R) \langle W(\hat{L}_0) \rangle$$

$$\langle W(\hat{L}_+) \rangle = \langle W(\hat{L}_0) \rangle \left[1 - \frac{2\pi i}{k} c_2(R) \right]$$

$\propto$

$$\langle W(\hat{L}_-) \rangle = \langle W(\hat{L}_0) \rangle \left[1 + \frac{2\pi i}{k} c_2(R) \right]$$

$\propto^{-1}$ (upto $\mathcal{O}(1/k^2)$)

It say.

$$\langle \delta U_g(x_1, x_2) \rangle = \frac{4\pi i}{k} C_2(R) \epsilon_{\alpha\beta\gamma} \langle U_g(x_1, x_2) \rangle \int_{\lambda_1}^{\lambda_2} \dot{x}^\beta(\lambda) \dot{x}^\gamma(\lambda) \delta x^\alpha(\lambda) d\lambda$$

in case of loops.

$$\langle \delta W_K \rangle = \frac{4\pi i}{k} C_2(R) \epsilon_{\alpha\beta\gamma} \langle W_K \rangle \oint_{S_1} \delta x^\alpha(\lambda) \dot{x}^\beta(\lambda) \dot{x}^\gamma(\lambda) d\lambda.$$

def: $v = \epsilon_{\alpha\beta\gamma} \oint_{S_1} \delta x^\alpha(\lambda) \dot{x}^\beta(\lambda) \dot{x}^\gamma(\lambda) d\lambda$

Σ .

* Point change $\delta x^\alpha(\lambda) \propto c^\alpha \delta(\lambda - \lambda^*)$

λ^* point of perturbation

$v = \epsilon_{\alpha\beta\gamma} c^\alpha \dot{x}^\beta(\lambda^*) \dot{x}^\gamma(\lambda^*)$

$v = \begin{matrix} +1/2 \\ -1/2 \\ 0 \end{matrix} \left. \begin{matrix} \text{out of plane.} \\ \text{if } c^\alpha \text{ lies in the plane defined by } \dot{x}^\beta(\lambda^*) \text{ \& } \dot{x}^\gamma(\lambda^*) \end{matrix} \right\}$

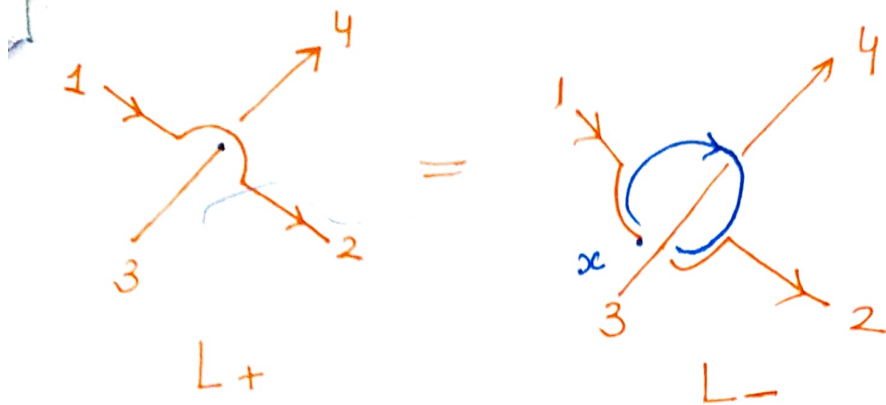
Case 1: $v = 0$

$\xrightarrow{\hat{L}_0} \leftrightarrow \xrightarrow{\Omega} \hat{L}_f$

$$\langle \delta(W_K) \rangle = 0$$

$$\langle W(\hat{L}_f) \rangle = \langle W(\hat{L}_0) \rangle$$

(ii)



using properties
of transport
& $\langle W(\hat{L}_f) \rangle = \langle W(\hat{L}_f^*) \rangle$

$$\langle W(L_+) \rangle = \langle \dots U(3,4) \dots U(x,2) U_c U(1,x) \dots \rangle$$

$$\downarrow$$

$$\left(1 - \sum_{\mu\nu} \frac{\mu\nu}{2} F_{\mu\nu}^a(x) T_a \right)$$

$$= \langle W(L_-) \rangle - \sum_{\mu\nu} \frac{\mu\nu}{2} \langle F_{\mu\nu}^a(x) \dots U(3,4) \dots U(x,2) T_a U(1,x) \dots \rangle$$

using 2

$$= \langle W(L_-) \rangle - \sum_{\mu\nu} \frac{\mu\nu}{2} \cdot \frac{4\pi i}{k} \epsilon_{\mu\nu\rho} g^{ab} \left(-\dot{x}^\rho(x^*) \right)$$

$$\left\langle \dots U(x,4)^{ij} T_b^{jk} U(3,x)^{kl} \dots \right.$$

$$\left. \dots U(x,2)^{ij} T_a^{jk} U(1,x)^{kl} \right\rangle$$

$$= \langle W(L_-) \rangle - \frac{4\pi i}{k} g^{ab} \left\langle \dots \right\rangle$$

$$- g^{ab} T_a^{ij} T_b^{kl} = \frac{1}{2} \delta^{ij} \delta^{kl} - \frac{1}{2N} \delta^{ik} \delta^{jl}$$

Fierz

$$N_L(\alpha, \beta, z) = \alpha^{-w(L)} S_L(\alpha, \beta, z)$$

$$N_0 = 1$$

$$w(\hat{L}_+) = 1 + w(\hat{L}_0)$$

$$N_{\hat{L}_+} = \alpha^{-w(\hat{L}_+)} S_{\hat{L}_+}$$

$$= N_{\hat{L}_0}$$

$$N_{\hat{L}_-} = N_{\hat{L}_0}$$

$$\begin{aligned} \underline{\text{Calc}} \quad (\alpha\beta) N_{L_+} - (\alpha\beta)^{-1} N_{L_-} &= (\alpha\beta) \alpha^{-w(L_+)} S_{L_+} \\ &\quad - (\alpha\beta)^{-1} \alpha^{-w(L_-)} S_{L_-} \end{aligned}$$

$$= \alpha^{-w(L_0)} (\beta S_{L_+} - \beta^{-1} S_{L_-})$$

$$= \alpha^{-w(L_0)} z S_{L_0}$$

$$= z N_{L_0}$$

N_k invariant under $\Omega_1, \Omega_2, \Omega_3$.

& $N_0 = 1$

$$\boxed{(\alpha\beta) N_{L_+} - (\alpha\beta)^{-1} N_{L_-} = z N_{L_0}}$$

Atomply for $\alpha\beta = t$

$$\alpha = 1 + i \frac{\pi}{k} \left(N - \frac{1}{N} \right) + \mathcal{O}(1/k^2)$$

$$\beta = 1 + i \frac{\pi}{k N} + \mathcal{O}(1/k^2)$$

$$Z = -i \frac{2\pi}{k} + \mathcal{O}(1/k^2)$$

$$\alpha\beta = t = 1 + \frac{i\pi N}{k}$$

$$= e^{\frac{i\pi N}{k}} + \mathcal{O}(1/k^2)$$

Witten's result

$$N=2$$

$$\text{Jones: } e^{\frac{2\pi i}{k}} = q$$

$$z = t - 1/t$$

APPENDIX A

SUPPLEMENTARY MATHEMATICAL PRELIMINARIES

A.1. Point-Set Topology

Definition A.1.1. — A **topology** on a set X is a subset $\tau \subseteq \mathcal{P}(X)$ of the powerset of X such that

- $\forall \mathcal{U} [\mathcal{U} \subseteq \tau \implies \bigcup \mathcal{U} \in \tau]$
- $\forall \mathcal{U} [\mathcal{U} \subseteq \tau \wedge \mathcal{U} \text{ is finite} \implies \bigcap \mathcal{U} \in \tau]$
- $\emptyset \in \tau$
- $X \in \tau$

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on it. In this case, those $U \in \tau$ are called **open** sets.

Definition A.1.2. — A map $f: X \rightarrow Y$ is **continuous** if for any open $U \subseteq Y$, we have $f^{-1}(U) \subseteq X$ is open. A continuous bijection with continuous inverse is called a **homeomorphism**. We represent the set of continuous functions from x to Y as $C^0(X, y)$.

An **embedding** $f: X \rightarrow Y$ is continuous, injective and yields a homeomorphism $X \cong f(X)$.

Definition A.1.3. — Let $f, g: X \rightarrow Y$ be continuous such that they are equal when restricted to $A \subseteq X$. Then we say they are homotopic rel A if there is a map $H: X \times I \rightarrow Y$ such that

- $H(-, 0) = f$
- $H(-, 1) = g$
- $H(a, t) = f(a) = g(a) \forall a \in A \text{ and } \forall t \in I.$

A homotopy rel $\emptyset$ is called a **homotopy**. A homotopy rel $\{a, b\}$, fixing two points is **path-homotopy**. If two maps f, g are homotopic rel $\emptyset$, we write $f \simeq g$.

We say two spaces X, Y are **homotopy equivalent** if there are maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$.

Definition A.1.4. — A collection of open sets $\mathcal{B}$ is called a **basis** for a topology if all open sets can be written as unions of sets from $\mathcal{B}$.

Some properties a space can have are:

- A space X is called **Hausdorff** if for any two points $x_1, x_2 \in X$ there are open sets $U_i \ni x_i$ such that $U_1 \cap U_2 = \emptyset$.
- A space is called **second countable** if it has a countable basis.
- A space is called **contractible** if it is homotopy equivalent to a single point.

A.2. Differential Topology

A **chart** on a space X is a pair (U, ϕ) where U is open and $\phi: U \rightarrow \mathbb{R}^n$ is a homeomorphism. Given two charts (ϕ_1, U_1) and (ϕ_2, U_2) , we say they are C^k -**compatible** if $\phi_1|_{U_1 \cap U_2} \circ \phi_2|_{U_1 \cap U_2}^{-1}: \phi_2(U_1 \cap U_2) \rightarrow \phi_1(U_1 \cap U_2)$ and the same expression with 1, 2 switched are C^k maps. Compatible maps always have the same codomain because they are invertible and the invariance of domain. A C^k -**atlas** is a collection of charts $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ such that any pair of charts in $\mathcal{A}$ are C^k -compatible and $\bigcup_{i \in I} U_i = X$ (we say that the charts cover X). A C^k -atlas is said to be **maximal** if any chart C^k -compatible with all the charts in the atlas is already in the atlas. Any C^k -atlas determines a unique maximal C^k -atlas as it can be constructed by just adding all the charts C^k -compatible with all the charts already in it. An atlas is called **oriented** if all the transition functions have positive Jacobian.

Definition A.2.1. — A C^k -**manifold** is a second countable, Hausdorff space with a maximal C^k -atlas. It is an (C^k, n) -**manifold** such that all the charts map open subsets of $\mathcal{M}$ into $\mathbb{R}^n$. It is called a **manifold with boundary** if some charts are onto $\mathbb{H}^n$. In this case, the **boundary** of the manifold is defined to be those points which have charts that map on $\mathbb{H}^n$ and when mapped there, they lie on the boundary.

A manifold is said to be **orientable** if we can give it an oriented atlas. We give $\mathbb{R}^n$ the obvious manifold structure by forming the maximal atlas that contains the identity map. Because $\mathbb{R}^n \not\cong_k \mathbb{R}^m$ for $n \neq m$ (this $\cong_k$ is through C^k bijection with C^k inverse), we have that the dimension of an (C^k, n) -manifold (denoted $\dim \mathcal{M} = n$) is a well defined expression.

Definition A.2.2. — A map $f: \mathcal{M} \rightarrow \mathcal{N}$ between a (C^k, m) -manifold and a (C^k, n) -manifold is said to be C^k if for any point p and a chart (U, ϕ) on $\mathcal{M}$

and (V, ψ) on $\mathcal{N}$ such that $p \in U$ and $f(p) \in V$ we have that the map

$$\psi|_{f(U \cap f^{-1}(V))} \circ f|_{U \cap f^{-1}(V)} \circ \phi|_{U \cap f^{-1}(V)}^{-1}$$

which maps $\phi(U \cap f^{-1}(V)) \subseteq \mathbb{R}^m$ to $\psi(f(U \cap f^{-1}(V))) \subseteq \mathbb{R}^n$ is C^k . A C^k -**diffeomorphism** is a C^k bijection with C^k inverse.

We denote the set of C^k function from $\mathcal{M}$ to $\mathcal{N}$ by $C^k(\mathcal{M}, \mathcal{N})$. Define the space of **locally defined functions at p** as

$$LC_p^k(\mathcal{M}, \mathcal{N}) = \bigsqcup_{p \in U \subseteq \mathcal{M} \text{ open}} C^k(U, \mathcal{N})$$

Definition A.2.3. — Let $\mathcal{M}$ be a (C^k, m) -manifold and $p \in \mathcal{M}$. Define an equivalence relation on $LC_p^k(\mathcal{M}, \mathcal{N})$ by $f \sim_p g$ if there is an open set $V \ni p$ in $\mathcal{M}$ such that $f|_V = g|_V$. Then the partition of $LC_p^k(\mathcal{M}, \mathcal{N})$ by $\sim_p$ is called the **stalk** of $C^k(\mathcal{M}, \mathcal{N})$ at p and is represented by $C_p^k(\mathcal{N})$. An element of the stalk (which is an equivalence class) is called a **germ** and is represented by $[f]_p$ where f is a representative of that class. We say that the germ of f at p is $[f]_p$.

We will denote $C_p^k(\mathbb{R})$ as C_p^k . It is easy to see that C_p^k carries the structure of an $\mathbb{R}$ -algebra.

Definition A.2.4. — A **tangent vector** at p is a map $v_p: C_p^k \rightarrow \mathbb{R}$ which is linear and satisfies the following version of the Leibnitz rule:

$$v_p([f]_p [g]_p) = f(p) v_p[g]_p + g(p) v_p[f]_p$$

This is well defined as functions in the same germ will take the same values at p . The set of all tangent vectors at a point is called the **tangent space** and is denoted by $T_p\mathcal{M}$.

Given a chart (U, ϕ) with $p \in U$ we can form the tangent vectors

$$\begin{aligned} \partial_i|_p: C_p^k &\rightarrow \mathbb{R} \\ [f]_p &\mapsto \frac{\partial}{\partial x^i} \left(f|_{U \cap \text{dom} f} \circ \phi|_{U \cap \text{dom} f}^{-1} \right) |_{\phi(p)} \end{aligned}$$

It is easy to see that $T_p\mathcal{M}$ is a vector space over $\mathbb{R}$ with basis $\partial_i|_p$. Hence it has dimension $\dim \mathcal{M}$. Also, any vector can be written as $v_p = v^i \partial_i|_p$. The v^i are called the components of v_p in coordinates ϕ .

Definition A.2.5. — The **tangent bundle** will be defined in three steps. First as a set, then as a space and then as a manifold. Let

$$T\mathcal{M} = \bigsqcup_{p \in \mathcal{M}} T_p\mathcal{M}$$

Now, there is an obvious surjection $\pi: T\mathcal{M} \rightarrow \mathcal{M}$ which associates a tangent vector to its basepoint. Given any chart (U, ϕ) on $\mathcal{M}$, we have the map

$$\begin{aligned} \tilde{\phi}: \pi^{-1}(U) &\rightarrow \mathbb{R}^{2n} \\ (p, v) &\mapsto (\phi(p), v^1, \dots, v^n) \end{aligned}$$

where $v = v^i \partial_i|_p$. It is easy to see that this map is bijective. We define a topology on $T\mathcal{M}$ by saying that $A \subseteq T\mathcal{M}$ is open if and only if for any chart (U, ϕ) , $\tilde{\phi}(A \cap \pi^{-1}(U))$ is open in $\mathbb{R}^{2n}$. Hence, by definition $\tilde{\phi}: \pi^{-1}(U) \rightarrow \mathbb{R}^{2n}$ is a homeomorphism and can hence serve as an atlas on $T\mathcal{M}$. Since the Jacobian of $\phi \circ \psi^{-1}$ is differentiable for $k > 1$, the transition maps $\tilde{\phi} \circ \tilde{\psi}^{-1}$ are also C^k . Hence these do form an atlas. We can construct the maximal atlas containing this atlas using the general construction. It is also easy to see that $\pi \in C^k(T\mathcal{M}, \mathcal{M})$. For any contractible manifold we can show that $T\mathcal{M} \cong \mathcal{M} \times \mathbb{R}^n$. Since there is a chart around every p and $\mathbb{R}^n$ is contractible, we have an open set U around every point such that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\sim} & U \times \mathbb{R}^n \\ \pi|_{\pi^{-1}(U)} \downarrow & \swarrow \text{proj}_1 & \\ U & & \end{array}$$

Putting all this together, we will later see that this information means that $(\pi: T\mathcal{M} \rightarrow \mathcal{M}, \mathbb{R}^{\dim \mathcal{M}})$ is a $\dim \mathcal{M}$ dimensional $\mathbb{R}$ vector bundle which is C^k if $\mathcal{M}$ is a C^k -manifold for $k > 1$. It is also easy to see that $\dim T\mathcal{M} = 2 \dim \mathcal{M}$. For each p , we call $\pi^{-1}(p) = T_p\mathcal{M}$ the fibre of $T\mathcal{M}$ above p .

Definition A.2.6. — Given $f \in C^k(\mathcal{M}, \mathcal{N})$, we define the **derivative** $df: T\mathcal{M} \rightarrow T\mathcal{N}$ by

$$\begin{aligned} df|_p: T_p\mathcal{M} &\rightarrow T_{f(p)}\mathcal{N} \\ (df|_p v_p) [g]_{f(p)} &= v_p [g \circ f|_{f^{-1}(\text{dom } g)}]_p \end{aligned}$$

We can easily see that $df|_p$ is linear. So we say that df is fibrewise-linear. It is also to show that given maps $\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P}$, then $d(g \circ f) = dg \circ df$. That is the map that sends manifolds to their tangent bundles and functions between them to their derivatives is a functor.

For maps $f: \mathcal{M} \rightarrow \mathbb{R}$, we overload this notation and define df as $df|_p v_p = v_p[f]_p$

Definition A.2.7. — An **immersion** is a smooth map $f: \mathcal{M} \rightarrow \mathcal{N}$ if $df|_p: T_p \mathcal{M} \rightarrow T_{f(p)} \mathcal{N}$ is injective $\forall p \in \mathcal{M}$. It is a **smooth embedding** if it is an embedding with is also an injective immersion.

Given two *smooth embeddings* $f, g: \mathcal{M} \rightarrow \mathcal{N}$ a homotopy $H: \mathcal{M} \times I \rightarrow \mathcal{N}$ is an **isotopy** if $H(-, t): \mathcal{M} \rightarrow \mathcal{N}$ is a *smooth embedding* $\forall t \in I$.

Definition A.2.8. — We define a **vector field** as C^k map $X: \mathcal{M} \rightarrow T\mathcal{M}$ such that $\pi \circ X = \text{id}_{\mathcal{M}}$. A locally defined vector field is a map $X: U \rightarrow T\mathcal{M}$ such that $\pi(X(p)) = p$ for $p \in U$. We denote the space of vector fields on open $U \subseteq \mathcal{M}$ by $\Gamma_U(T\mathcal{M})$. The space of globally defined vector fields is denoted $\Gamma(T\mathcal{M})$. It is easy to see that $\Gamma_U(T\mathcal{M})$ is a vector space over $\mathbb{R}$ and a module over $C^k(U)$.

Given any vector field $X \in \Gamma_U(T\mathcal{M})$ and $f \in C^k(U)$, define $(Xf)(p) = X_p[f]_p$. Thus, a vector field can also be thought of as a map from $X: C^k(U) \rightarrow C^{k-1}(U)$.

Definition A.2.9. — Given two locally defined vector fields $X, Y: C^\infty(U) \rightarrow C^\infty(U)$, we can see that $[X, Y] = X \circ Y - Y \circ X$ is also a vector field. This is called the **commutator**. In coordinates, it is given by

$$[X, Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i$$

It is again easy to see that the commutator turns $\Gamma_U(T\mathcal{M})$ into an infinite dimensional Lie algebra.

Given a diffeomorphism F , we define $F_*: \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$ by

$$(F_*X)_p[f]_p = X_{F^{-1}(p)}[f \circ F]_{F^{-1}(p)}$$

It is easy to see that F_* is linear and it preserves the commutator, i.e. $[F_*X, F_*Y] = F_*[X, Y]$. Thus, the set of globally defined vector spaces preserved by F_* will also be a Lie algebra. Intersecting, given a family of diffeomorphisms F_i , the set of vector fields preserved by all of them forms a Lie algebra.

Definition A.2.10. — A **cotangent vector** at p is a linear map $\omega_p: T_p \mathcal{M} \rightarrow \mathbb{R}$. The set of all cotangent vectors at a point p represented by $T_p^* \mathcal{M}$ as it is the dual space to $T_p \mathcal{M}$. The **cotangent bundle** $T^* \mathcal{M}$ can be defined in a way similar to $T\mathcal{M}$. A local section of this bundle is called a **locally defined covector field**.

Since we have

$$\begin{array}{ccc}
 & \text{id}_{\mathbf{fdVect}_K} & \\
 \swarrow & \downarrow \sim & \searrow \\
 \mathbf{fdVect}_K & & \mathbf{fdVect}_K \\
 \nwarrow & \downarrow (-)^{**} & \nearrow
 \end{array}$$

We can think of a tangent vector $v_p: C_p^\infty \rightarrow \mathbb{R}$ as a linear map $v_p: T_p^* \mathcal{M} \rightarrow \mathbb{R}$. Given a covector field $\omega: T\mathcal{M} \rightarrow \mathbb{R}$ and map $f: \mathcal{N} \rightarrow \mathcal{M}$, then we have the **pullback** defined as

$$f^* \omega = \omega \circ df$$

Definition A.2.11. — A **rank (k, l) -tensor** at p is a multilinear map

$$L_p: \left(\bigotimes_{i=1}^k T_p^* \mathcal{M} \right) \otimes \left(\bigotimes_{j=1}^l T_p \mathcal{M} \right) \rightarrow \mathbb{R}$$

The collector of (k, l) -tensors at a point p is represented by $T_{pl}^k \mathcal{M}$ and the union is called the $T_l^k \mathcal{M}$. A section is a locally defined rank (k, l) tensor field. There is the obvious way to add and take tensor products of tensors. This turns $\mathcal{T}_p \mathcal{M} = \bigoplus_{k,l} T_{pl}^k$ into a $\mathbb{N}^2$ -graded $\mathbb{R}$ -algebra.

Definition A.2.12. — A k -form is rank $(0, k)$ tensor which is completely antisymmetric. The set of k -forms at p is represented by $\Lambda^k T_p^* \mathcal{M}$. The union of these over p is represented by $\Lambda^k T^* \mathcal{M}$. We can define the exterior product of a k -form α and a l -form β as

$$\alpha \wedge \beta = \frac{(k+l)!}{2k!l!} (\alpha \otimes \beta - \beta \otimes \alpha)$$

This is an antisymmetric product and gives out at $k+l$ -form. Thus this turns $\Lambda T_p^* \mathcal{M}$ (union over k) into an $\mathbb{N}$ -graded antisymmetric $\mathbb{R}$ -algebra. We define the **exterior derivative** as a map $d: \Gamma_U(\Lambda T^* \mathcal{M}) \rightarrow \Gamma_U(\Lambda T^* \mathcal{M})$ to be the unique map satisfying the following properties

- Linearity
- Maps k -forms to $k+1$ -forms
- For 0-forms, $df(X) = Xf$
- For 0-forms $d^2 f = 0$
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$

In coordinates, if α is a k -form with representation

$$\alpha = \frac{1}{k!} \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

then

$$d\alpha = \frac{1}{k!} \frac{\partial}{\partial x^{i_0}} (\alpha_{i_1 \dots i_k}) dx^{i_0} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Definition A.2.13. — A **smooth partition of unity** subordinate to an open cover $\{U_\alpha\}_{\alpha \in A}$ of a smooth manifold $\mathcal{M}$ is a collection of smooth maps $\psi_\alpha: \mathcal{M} \rightarrow \mathbb{R}$ such that

- $0 \leq \psi_\alpha(p) \leq 1$
- $\text{supp } \psi_\alpha \subseteq U_\alpha$ where $\text{supp } \psi_\alpha = \overline{\mathcal{M} \setminus \psi_\alpha^{-1}(0)}$
- The family of supports $\{\text{supp } \psi_\alpha\}_{\alpha \in A}$ is **locally finite**. That is, for any point $p \in \mathcal{M}$, it has an open neighbourhood which intersects only finitely many of the supports.
- $\sum_{\alpha \in A} \psi_\alpha(p) = 1$

A smooth partition of unity always exists on a smooth manifold.

Let ω be a n -form on an open subset $U \subseteq \mathbb{R}^n$ or $U \subseteq \mathbb{H}^n$. Since it is a top dimensional form, we can write it as $\omega = f dx^1 \wedge \dots \wedge dx^n$. We define

$$\int_D \omega = \int_D f dV$$

where $D \subseteq U$ is a Borel-measurable set and dV is the Lebesgue measure on $\mathbb{R}^n$. Let $\mathcal{M}$ be an oriented smooth n -manifold and let ω be an n -form on it which is supported in a single chart. Then we define

$$\int_{\mathcal{M}} \omega = \pm \int_{\phi(U)} (\phi^{-1})^* \omega$$

where we take $+$ if the chart is oriented positively (in the oriented atlas) and $-$ otherwise. Now we let ω be an arbitrary n -form on an n -manifold. Let $\{U_\alpha\}$ be a cover of the support of ω by charts, and let ψ_α be a corresponding partition of unity. Then we define

$$\int_{\mathcal{M}} \omega = \sum_{\alpha} \int_{\mathcal{M}} \psi_\alpha \omega$$

Each integral in the sum is defined as $\psi_\alpha \omega$ is supported in the chart U_α .

Definition A.2.14. — **Generalised Stokes' Theorem:** Let ω be a smooth $n-1$ -form with compact support on a smooth oriented n -dimensional manifold-with-boundary Ω , and let $\partial\Omega$ be the boundary of Ω with the induced orientation, and let $\iota: \partial\Omega \rightarrow \Omega$ be the inclusion map, then

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \iota^* \omega$$

A.3. Algebra

Definition A.3.1. — A **group** is a set G equipped with a specified element $e \in G$, a unary operation $(-)^{-1} : G \rightarrow G$ and a binary operation $\cdot : G \times G \rightarrow G$ which satisfy:

- Associativity: $\forall g_1, g_2, g_3 \in G [g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3]$
- Identity: $\forall g \in G [g \cdot e = e \cdot g = g]$
- Inverse: $\forall g \in G [g \cdot g^{-1} = e = g^{-1} \cdot g]$

We will symbolize group multiplication by concatenation (omitting the $\cdot$ notation).

A **semi-group** is a set with a multiplication binary map which only satisfies the first group axiom, i.e. associativity.

A **monoid** is a set with appropriate structure and a multiplication map defined on it, which satisfies only first two of the group axioms (associativity and existence of identity) and does not have an inverse for each and every element under the multiplication operation.

An **abelian group** is a group whose multiplication operation is commutative, i.e. $\forall g_1, g_2 \in G, g_1 g_2 = g_2 g_1$

A **subgroup** $H \subseteq G$ is a subset of G which contains the identity and is closed under multiplication.

Definition A.3.2. — A **group homomorphism** is a map $h : G \rightarrow H$ which satisfies:

- $\forall g_1, g_2 \in G [h(g_1 g_2) = h(g_1) h(g_2)]$ with relevant notion of group multiplication. This property leads to the following
 - $h(e) = e$
 - $h(g^{-1}) = h(g)^{-1}$

A **group isomorphism** is a group homomorphism with an inverse, in other words it is a bijective homomorphism. We denote the set of isomorphisms of a group with itself as $\text{Aut}G$. A self isomorphism is called an **automorphism**.

Definition A.3.3. — A **left group action** of a group G on a set X (denoted by $G \curvearrowright X$) is a map $\phi : G \times X \rightarrow X$ such that

- $\phi(e, x) = x$
- $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$

We will often denote $\phi(g, x) = g \cdot x$. Fixing an element $g \in G$ we get a map $\phi_g : X \rightarrow X$ which is a bijection. In fact, we can also think of the action as a homomorphism $G \rightarrow \text{Aut}X$ where the automorphisms of a set are bijections to itself. A right group action can be defined similarly. The **orbit** of a point under an action is the set of points it reaches when acted upon by the entire group. The set of orbits is denoted by X/G . An action is said to be **free** if for any $g \cdot (-) : X \rightarrow X$ has a fixed point, then $g = e$. An action is said to

be **transitive** if every orbit is the entire space. Specifically, transitivity is the existence and freeness is the uniqueness of group elements when they try to shift a point to the other. The set of orbits is called the orbit space X/G .

Definition A.3.4. — A representation of a group is an action on a vector space that is linear. That is, it is a homomorphism $\rho: G \rightarrow \text{Aut } V$. We call a representation **trivial** if $\rho(g) = \text{id}_V$ for all g .

Given a representation $\rho: G \rightarrow \text{Aut } V$, we can form the **dual representation** as

$$\begin{aligned}\rho^*: G &\rightarrow \text{Aut } V^* \\ g &\mapsto \rho(g^{-1})^*\end{aligned}$$

Given two representations $\rho_i: G \rightarrow \text{Aut } V_i$ then we define

$$\begin{aligned}\rho_1 \oplus \rho_2: G &\rightarrow \text{Aut } (V_1 \oplus V_2) \\ \rho_1 \oplus \rho_2(g)(v_1 \oplus v_2) &= \rho_1(g)v_1 \oplus \rho_2(g)v_2\end{aligned}$$

and

$$\begin{aligned}\rho_1 \otimes \rho_2: G &\rightarrow \text{Aut } (V_1 \otimes V_2) \\ \rho_1 \otimes \rho_2(g)(v_1 \otimes v_2) &= \rho_1(g)v_1 \otimes \rho_2(g)v_2\end{aligned}$$

Given a representation ρ of G on a real vector space, we define its **complexification** as

$$\begin{aligned}\rho_{\mathbb{C}}: G &\rightarrow \text{Aut } (V \otimes_{\mathbb{R}} \mathbb{C}) \\ \rho_{\mathbb{C}} &= \rho \otimes t\end{aligned}$$

where t is the trivial representation of G on $\mathbb{C}$.

Definition A.3.5. — A **ring** R is a set with a distinguished element 0, a unary operation $-: R \rightarrow R$ and two binary operations $+, \cdot$ from $R \times R \rightarrow R$ such that

- $(R, +, -, 0)$ is an abelian group
- $(R, \cdot)$ is a semigroup
- $\cdot$ distributes over $+$, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$

We will typically leave out $\cdot$ and represent by concatenation. A ring can also be thought of as a semigroup acting on an abelian group. We call a ring commutative if multiplication is **commutative** and we call it **unital** if there is a multiplicative identity. A **field** is a commutative, unital ring such that every non-zero element has a multiplicative inverse.

Definition A.3.6. — An **left R -module** (or module over a ring R) is an abelian group M and a map $\cdot : R \times M \rightarrow M$ such that

$$\begin{aligned} r \cdot (x + y) &= r \cdot x + r \cdot y \\ (r + s) \cdot x &= r \cdot x + s \cdot x \\ r \cdot (s \cdot x) &= (rs) \cdot x \end{aligned}$$

It is called a **unital left module** if the ring is unital and $1 \cdot x = x$. A right module can be defined similarly. Left and right modules are the same if the ring is commutative. Hence for a commutative ring R , we just speak of R -modules. A unital module over a field is called a **vector space** over the field.

Definition A.3.7. — A **Lie Algebra** is a vector space together with a map $[-, -] : V \times V \rightarrow V$ such that it is bilinear, antisymmetric and satisfies the Jacobi identity. This map is called a Lie bracket. The Jacobi identity is

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

A.4. Lie Theory

Definition A.4.1. — A **Lie Group** G is a C^k manifold which at the same time is a group and the group structure is compatible with the manifold structure, i.e. the following maps are C^k differentiable

- Group multiplication $G \times G \rightarrow G, (g_1, g_2) \mapsto g_1 g_2$
- Inverse $G \rightarrow G, g \mapsto g^{-1}$

Let G be a Lie group, let $H \subset G$ be a subgroup (algebraically) and closed as a subset. Then H is a submanifold and a Lie group in its own right.

Definition A.4.2. — A **Lie group homomorphism** is a C^k group homomorphism. A **Lie group isomorphism** is a Lie group homomorphism whose inverse exists and whose inverse is itself a Lie group homomorphism.

The importance of the concept of Lie algebra is that there is a special finite dimensional Lie algebra intimately associated with each Lie group, and that properties of the Lie group are reflected in properties of its Lie algebra. We shall study the construction of the Lie algebra from a Lie Group.

Let $L_g : G \rightarrow G$ be a G acting on itself by left multiplication. Then the vector fields such that $(L_g)_* X = X$ form a Lie algebra. These are called left invariant vector fields and form a Lie algebra. They satisfy

$$dL_g|_h X_h = X_{gh}$$

Thus, $dL_g|_e X_e = X_g$. So a left invariant vector field is completely determined by its value at e . This gives us an vector space isomorphism between left invariant vector fields and $T_e G$. We use this to import the Lie algebra structure onto

$T_e G$. We call this Lie Algebra (or the Lie algebra of left-invariant vector fields) the **Lie algebra corresponding to G** and write it as $\text{Lie } G$. We will shift between the definitions as the space of left invariant vector fields and as the tangent space at e freely. In particular, $\dim \mathfrak{g}$ (as real vector space) equals $\dim G$ (as smooth manifold).

Definition A.4.3. — Let $a: \mathcal{M} \times G \rightarrow \mathcal{M}$ be a differentiable action of a Lie group G on a manifold $\mathcal{M}$. For $p \in \mathcal{M}$, define $a_p = a(p, -): G \rightarrow \mathcal{M}$. Thus, we have

$$da_p|_e: \mathfrak{g} \rightarrow T_p \mathcal{M}$$

We define the **fundamental vector field** corresponding to $X \in \mathfrak{g}$ as $X^\#(p) = da_p|_e X_e$.

Definition A.4.4. — A **Lie Group representation** is a Lie group homomorphism map $\rho: G \rightarrow \text{Aut}(V)$ where V is some vector space over $\mathbb{R}$ or $\mathbb{C}$. A representation is said to be **faithful** if it is injective.

Let $c_g: G \rightarrow G$ be the conjugation, that is, $c_g(h) = ghg^{-1}$. Differentiating at identity, we get $dc_g|_e: \mathfrak{g} \rightarrow \mathfrak{g}$. We call the map

$$\begin{aligned} \text{Ad}: G &\rightarrow \text{Aut}(\mathfrak{g}) \\ g &\mapsto dc_g|_e \end{aligned}$$

It is obvious that Ad is smooth as α depends smoothly on g , hence so will its derivative. Now, we show it is a group homomorphism.

$$\begin{aligned} \text{Ad}_{gh} &= dc_{gh}|_e = d(c_g \circ c_h)|_e = dc_g|_{c_h(e)} \circ dc_h|_e \\ &= dc_g|_e \circ dc_h|_e = \text{Ad}_g \circ \text{Ad}_h \end{aligned}$$

Ad is called the **Adjoint representation**.

A.5. Seifert Surfaces of Links

Definition A.5.1. — A **Seifert Surface** for an oriented link L in $\mathbb{R}^3$ is a connected, compact, oriented surface embedded in $\mathbb{R}^3$ such that it has the link L as its oriented boundary.

An algorithm to get a Seifert Surface for a given diagram of the knot/ link is as follows.

- Pick an orientation for the diagram of the link.
- Remove the crossings by joining each incoming strand to the adjacent outgoing strand, creating a finite number of circles, called *Seifert circles*.

- Attach a disk to each Seifert circle, so that each Seifert circle now bounds a disk. In the case of concentric circles, vary the height of the disks so that the innermost Seifert circles are the highest. Chose a convention for the orientation, like anti-clockwise orientation at the boundary has its normal outwards and clockwise has its normal inwards. (they can also be distinguished by shading opposite oriented disks differently.
- At the site of each of the old crossings, attach the 2 neighboring disks by gluing in a rectangular band with a twist, the direction of the twist being determined by the original crossing and merge the disks with the bands.

The resultant surface is a Seifert surface.

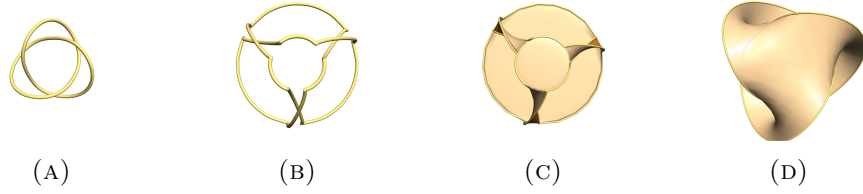


FIGURE 1. Seifert Surface Construction of Trefoil Knot

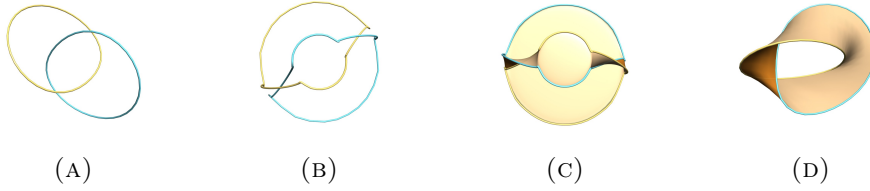


FIGURE 2. Seifert Surface Construction of the Hopf Link

Definition A.5.2. — The genus of a link L , denoted $g(L)$, is the minimum genus of all Seifert surfaces for L . The algorithm does not always give the surface with minimum genus so all we can claim is that $g(L) \leq g_s$, where g_s is the genus of the seifert surface constructed.

The genus of the a given surface can be calculated by the formula

$$g_s = \frac{2 + c - s - n}{2}$$

, where c is the number of crossings in the diagram of the link being considered, s is the number of seifert circles produced in the construction, n is the number of components of the link (1 for knots). This result follows trivially from considering simplicial complexes, i.e. by finding the euler characteristic of the surface by triangulization.

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