Gauge Theory and Knots

A mathematical bridge between Chern Simons Theory and the Knot Polynomials

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December 7, 2019

Contents

Bundle theoretic picture of a Gauge Theory

Fibre Bundles

Let E, B and F be smooth manifolds and $\pi\colon E \twoheadrightarrow B$ be a surjective smooth map. Then (E, B, π, F) is a **smooth fibre bundle** iff

- $\forall b \in B$, $\exists$ open $U \subseteq B$ containing b and $\exists$ diffeomorphism $\psi_U\colon \pi^{-1}(U) \xrightarrow{\sim} U \times F$ such that

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\psi_U", "\sim"'] \arrow[d, two heads, "\pi|_{\pi^{-1}(U)}"'] & U \times F \arrow[dl, two heads, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}$$

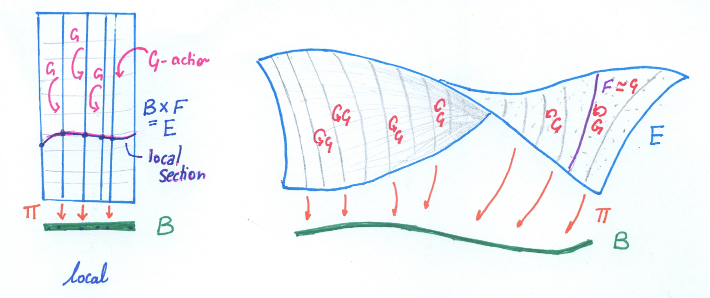
Principal G-bundles

Let (E, B, π, F) be a smooth fibre bundle and $\left(G, \mu, e, (-)^{-1}\right)$ be a Lie group. Let G act on E smoothly from the right by $a\colon E \times G \to E$, $a(p, g) = p \cdot g$. Then (E, B, π, F, a) is a **principal *G*-bundle** iff

- a acts freely and transitively on fibres
- $\forall p \in E$, $a_p\colon G \to \pi^{-1}(\{p\})$ (defined by $a_p(g) = p \cdot g$) is a diffeomorphism
- By the above point $F \simeq G$ (as manifolds), hence we have the local trivialisation $\psi_U\colon \pi^{-1}(U) \to U \times G$. We require them to also satisfy

$$\psi(p \cdot g) = \psi(p) \cdot g$$

where G acts on $U \times G$ by multipliying the second component from the right.



Gauge Transformation

- Let (E, B, π, G) and (E', B, π', G) be principal G-bundles. A map $f\colon E \to E'$ is a morphism of principal G-bundles if it is compatible with the action of G, i.e.

$$f(p \cdot g) = f(p) \cdot g$$

- This implies that $f\colon E \to E'$ gives rise to a morphism of bases spaces on composition with the map π', as it preserves the fibre structure (maps fibres to fibres). Let this induced map $\bar{f}\colon B \to B'$.
- If f is a diffeomorphism $E \to E$ and a morphism of G-bundles, then we call it an automorphism.

Gauge Transformations

An automorphism of E that induces id_B on its base space, i.e.

$$\bar{f}\colon B \to B = \mathrm{id}_B$$

is called a **gauge transformation**.

Connection

A **connection form** on a G-bundle is a $\mathfrak{g}$-valued 1-form ω such that

- For any $X_e \in \mathfrak{g}$, we have $\omega_p\left(X_p^{\#}\right) = X_e$
- For any $g \in G$, we have $a_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$

where $\mathfrak{g}$ is the Lie-Algebra of the Lie Group G;
$f^*\omega = \omega \circ \mathrm{d}f$, for a covector field ω on the manifold $\mathcal{M}$ and a function $f \colon \mathcal{N} \to \mathcal{M}$;
$X^{\#}$ is the fundamental vector field defined as $X_p^{\#} = \mathrm{d}a_p|_e X_e$

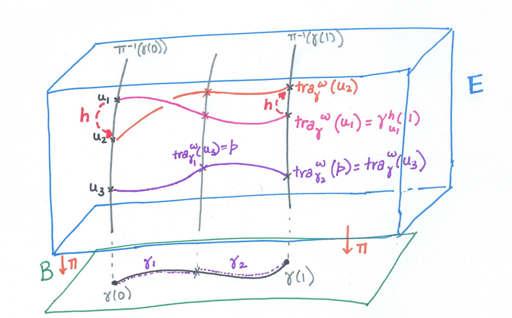
- We can write $v \in T_uE$ as $v = X_h + X_u^{\#}$ as the tangent space is the sum of horizontal and vertical spaces. The vertical space consists of fundamental fields evaluated at u, i.e. tangents to the fibre and the horizontal space is defined as $\ker\omega$.
- If $f \colon E' \to E$ is a morphism of principal G-bundles and ω is the connection-1-form on E, then $f^*\omega \colon TE' \to \mathfrak{g}$ is also a connection form *pulled back* from E to E' through the map f. We shall use this fact for the special case when f is a gauge transformation.

Parallel Transport

If $(\pi\colon E \to B, F, a)$ be a principle G-bundle with a connection form ω. For a piecewise smooth path $\gamma\colon I \to B$ and a $u_0 \in \pi^{-1}(\{\gamma(0)\})$ there is a unique path $\gamma^{\mathrm{h}}\colon I \to E$ such that

- Piecewise smooth: $\gamma^{\mathrm{h}}_{u_0}$ is smooth whenever γ is.
- Lift: $\pi \circ \gamma^{\mathrm{h}}_{u_0} = \gamma$
- Begins at u_0: $\gamma^{\mathrm{h}}_{u_0}(0) = u_0$
- Horizontal: $\omega_{\gamma^{\mathrm{h}}_{u_0}(t)}\left(\mathrm{d}\gamma^{\mathrm{h}}_{u_0}|_t(1)\right) = 0$ whenever defined.

Therefore, we define **parallel transport** as $tra^{\omega}_{\gamma}(u) = \gamma^{h}_{u}(1)$



Wilson Loop

- Let $\gamma_b\colon I \to B$ such that $\gamma(0) = \gamma(1) = b$, then the parallel transport map

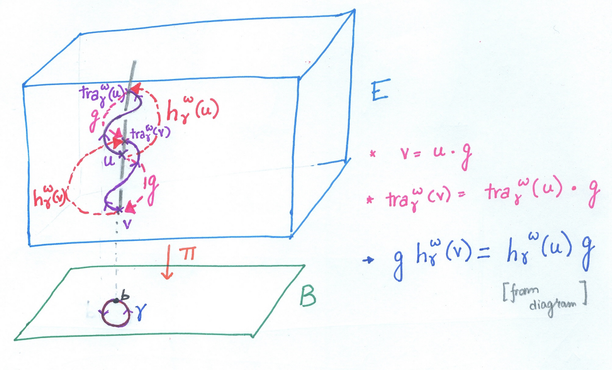
$$\mathrm{tra}^{\omega}_{\gamma_b}(u) = u \cdot h^{\omega}_{\gamma_b}(u)$$

can be used to define a new map $h^{\omega}_{\gamma_b}\colon \pi^{-1}(u) \to G$ for $u \in \pi^{-1}(b)$.
- It is shown that for different choice of points on the fibre above b we get conjugate group elements. Thus, we can define a map which takes a closed curve in the B and gives a specific conjugacy class of G.
- Now, given a map $K\colon S^1 \to B$, we have a choice of a starting point, say b to turn it into $\gamma_b\colon I \to B$. It is shown that for different choices of the base point we get the same conjugacy class.

We define the **Wilson Loop** of a curve in the representation ρ as the trace of the representation of any element from its conjugacy class. Specifically,

$$W_K(\omega) = \mathrm{Tr}_{\rho}\left(h^{\omega}_{\gamma_b}(u)\right) = \mathrm{Tr}_{\rho}\left(h^{\omega}_{\gamma_c}(v)\right)$$

Wilson Loop cont.



The Wilson Loop is gauge invariant.

If $f\colon E \to E$ is a gauge transformation and $f^*\omega$ is the pullback of ω under f, then using the properties of the Wilson Loop, it is shown that

$$W_K(\omega) = W_K(f^*\omega)$$

Trivial Bundle Case

The trivial bundle is described as,

$$F = G \qquad E = B \times G \qquad \pi(b,g) = b \qquad (b,g)\cdot h = (b,gh)$$

It is shown that

- A gauge transformation f can be written as

$$f(b,g) = (b, \alpha(b)\, g)$$

  for some $\alpha\colon B \to G$. This α is unique.
- The connection form form $\omega\colon T(B \times G) \to \mathfrak{g}$ can be written as

$$\omega_{b,g} u_{(b,g)} = \mathrm{Ad}_{g^{-1}} \circ A_b v_b + \theta_g w_g$$

  for some form $A\colon TB \to \mathfrak{g}$ and $\mathrm{d}L_{g^{-1}}|_g = \theta_g$ is the Maurer-Cartan form. The function A is unique and hence we shall work with A from further on.
- Under gauge transformation f characterized by the map α the function A characterizing the connection form transforms as, $\tilde{\omega} = f^*\omega$, which in terms of A can be written as

$$\tilde{A}_b = \mathrm{Ad}_{\alpha(b)^{-1}} \circ A_b + \mathrm{d}\left(L_{\alpha(b)^{-1}} \circ \alpha\right)|_b = \mathrm{Ad}_{\alpha(b)^{-1}} \circ A_b + (\alpha^*\theta)_b$$

Trivial Bundle with Matrix Group

In the special case were the Lie Group G is a matrix group

- The gauge transformation can be written in coordinates as

$$\tilde{A}_\mu (x^\nu) = \alpha (x^\nu)^{-1} A_\mu (x) \alpha (x^\nu) + \alpha (x^\nu)^{-1} \partial_\mu \alpha (x^\nu)$$

- The Wilson Loop reduces to solving a differential equation for the group element by which the initial point changes. This is given as a Path-ordered Exponential

$$W = \mathrm{Tr}\,(g\,(1)) = \mathrm{Tr}\left(\mathcal{P}exp\left(-\int_0^1 A_{\gamma(t)}\,(\dot{\gamma}\,(t))\,dt\right)\right) = \mathrm{Tr}\left(\mathcal{P}exp\left(-\int_\gamma A\right)\right)$$

Chern-Simons Form

Construction of the Chern Simons Form

- The Curvature F is defined as $F = dA + A \wedge A$.
- Under a gauge transformation, F transforms as $F \rightarrow \alpha^{-1} F \alpha$.
- From this we can define an interesting $2n-$form $\mathrm{Tr}\left(F^n\right) \equiv \mathrm{Tr}\left(F \wedge ... \wedge F\right)$ which is invariant under gauge transformations. We call this form the nth Chern form, denoted as $P_{2n}\left(F\right)$.
- $dP_{2n}\left(F\right) = 0$. We can locally express it as a derivative of a $(2n-1)$-form, $P_{2n}\left(F\right) = dQ_{2n-1}\left(a, F\right)$, This is the Chern-Simons (2n-1) form.
- $$\mathrm{Tr}\left(F \wedge F\right) = d\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Knots, Links and Invariants

Knots and Links

- A **knot** is an *embedding* of the circle S^1 into the Euclidean space $\mathbb{R}^3$, or equivalently into the 3-sphere S^3 (one can also consider knots in general 3-manifolds).
- A link is a *smooth-embedding* of $S^1 \bigsqcup S^1 \cdots \bigsqcup S^1 \to \mathbb{R}^3$ where $S^1 \bigsqcup S^1 \cdots \bigsqcup S^1$ denotes a disjoint union of several circles.
- Two knots are said to be isotopic (of the same isotopy type or equivalent) if one can be transformed into another by means of a *smooth isotopy*.
- A knot invariant is map from isotopy equivalence classes of knots to any kind of mathematical structure.

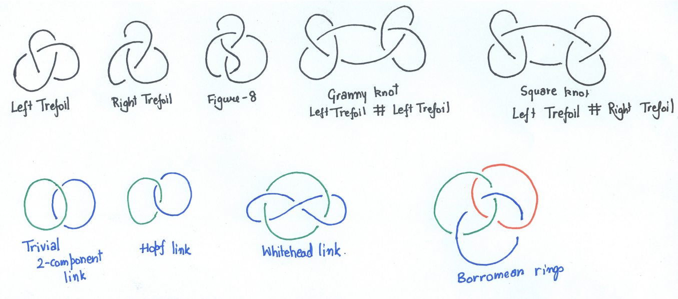
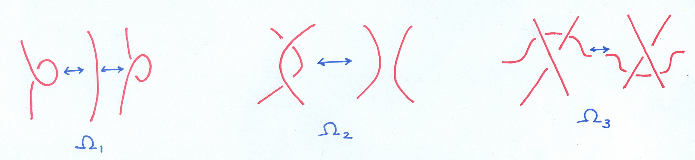


Figure: Knots and Links Diagrams

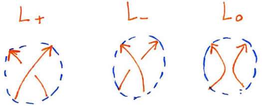
R-Moves and the HOMFLY Polynomial

Reidemeister: Two unoirented knots K_1 and K_2 are isotopic iff we can get from a diagram of the first to a diagram of the second by finite sequence of the following moves:



The HOMFLY-PT polynomial is an oriented links invariant polynomial (invariant under the three R-moves, and hence invariant under isotopy) in two variables t and z defined as

- The normalization $P(\mathrm{O}) = 1$
- The Skein Realtion $tP(L_+) - t^{-1}P(L_-) = zP(L_0)$



Path Integral to Link Polynomials

Structure of QFT

- The starting point for defining a quantum field theory is the choice of a base manifold B of dimension n.
- The next is the choice of objects over B.
- Further, one aims to integrate over the space parametrizing these objects (some kind of a moduli space).

- These objects defined on the base manifold are called fields.
- The operation of integration over the space of fields is also called the path-integral.
- In integrating over the field space we have to choose a measure on it. The measure is also usually weighted by $\exp(iS)$, where S is a functional on the space of fields in question and is called the action.

Quantum field theories associated with connections and a notion of guage transformation are called gauge theories. Here, the path-integral is over the space of gauge equivalence classes of connections (the space of connections modulo gauge transformations).

- The choice of a point from each orbit of the gauge transformations is called a gauge fixing condition.

Motivation for Looking for Invariants

- The basic strategy to get an invariant out of a QFT goes hand-in-hand with the basic crux of doing the path integral
- We endow the space with extra geometric structures (such as a connection), then compute a number from this manifold-with-connection (using the action functional)
- Finally we sum over all geometric structures (in this case, equivalence classes of connections) to take away the dependence on the added structure

The final result is a topological invariant because all the non-topological (geometric) data has been integrated over.
We use the Chern- Simoms action because it does not have any dependence on a metric and the connection is the only geometric quantity is the entire structure which we later perform the path integral over.

Path Integral in Chern Simons Theory

We take the action to be the Chern-Simons form

$$S_{\mathrm{CS}}[A] = \frac{1}{4\pi}\int_B \mathrm{d}A \wedge A + \frac{2}{3} A \wedge A \wedge A$$

The path integral is given by

$$Z_B(k) = \int_{\frac{\mathcal{C}}{\mathcal{G}}} \mathcal{D}A \; e^{ikS_{\mathrm{CS}}[A]} = \frac{1}{\mathrm{Vol}(\mathcal{G})}\int_{\mathcal{C}} \mathcal{D}A \; e^{ikS_{\mathrm{CS}}[A]}$$

where, $\mathrm{Vol}(\mathcal{G})$ is the volume of the infinite-dimensional group of gauge transformations, $\mathcal{C}$ is the space of connection formsf , $\mathcal{D}A$ is the measure in the space of all connections, B is the base manifold and $S_{CS}[A]$ is the action.

$$P_K = \int_{\frac{\mathcal{C}}{\mathcal{G}}} \mathcal{D}A \; \mathrm{Tr}_\rho \left(\mathrm{P}\exp\left(\oint_K A\right)\right) \exp\left(\frac{ik}{4\pi}\int_{\mathbb{R}^3} \mathrm{Tr}_\rho \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right)\right)$$

Variation of the Parallel Transport wrt. the curve

Let $\gamma \colon S^1 \to \mathbb{R}^3$ be an injective smooth map. Let $\lambda_1, \lambda_2 \in S^1$ with $\gamma(\lambda) = x^\mu(\lambda)$ in coordinates. Let $x(\lambda_i) = x_i$. Then define

$$U_\gamma(x_1, x_2) = \mathcal{P}\left(\exp\left(-\int_{\gamma|_{[\lambda_1,\lambda_2]}} A\right)\right)$$

Then we have

$$\frac{\delta U_\gamma(x_1, x_2)}{\delta x^\alpha(\lambda)} = \begin{cases} -\dot{x}^\beta(\lambda) F_{\alpha\beta}{}^a U_\gamma(\gamma(\lambda), x_2) T_a U_\gamma(x_1, \gamma(\lambda)) & \text{if } \lambda \in [\lambda_1, \lambda_2] \\ 0 & \text{otherwise} \end{cases}$$

This equation is valid for any parallel transport, i.e., it doesn't depend on the action.

Expectation Value of Curvature times an Operator

Let O be any operator. Then,

$$\left\langle F_{\mu\nu}{}^{a}(x)\, O \right\rangle = \frac{4\pi i}{k} \epsilon_{\mu\nu\rho} \delta^{ab} \left\langle \frac{\delta O}{\delta A_{\rho}{}^{b}(x)} \right\rangle$$

This is valid only in Chern-Simons theory, and this is the only place where Chern-Simons theory actually comes into play. We use

$$\epsilon_{\mu\nu\rho} \frac{\delta S_{CS}[A]}{\delta A_{\rho}{}^{b}} = \frac{k}{4\pi} F_{\mu\nu}{}^{a} \delta_{ab}$$

and a by-parts in the path integral.

Expectation Value of Variation of the Wilson Loop wrt. the curve

Using the previous two equations, we get

$$\left\langle \frac{\delta U_\gamma(x_1,x_2)}{\delta x^\alpha(\lambda)} \right\rangle = -\frac{4\pi i}{k}\dot{x}^\beta(\lambda)\,\delta^{ab}\epsilon_{\mu\nu\rho}\left\langle \frac{\delta}{\delta A_\rho{}^b(x^\sigma(\lambda))}\left(U_\gamma(\gamma(\lambda),x_2)\,T_a U_\gamma(x_1,\gamma(\lambda))\right)\right\rangle$$

Using

$$\frac{\delta U_\gamma(\gamma(\lambda),x_2)}{\delta A_\rho{}^b(x^\sigma(\lambda))} = -\dot{x}^\rho(\lambda)\,U_\gamma(\gamma(\lambda),x_2)\,T_b \quad \text{and} \quad \frac{\delta U_\gamma(x_1,\gamma(\lambda))}{\delta A_\rho{}^b(x^\sigma(\lambda))} = 0$$

We get

$$\frac{\delta U_\gamma(x_1,x_2)}{\delta A_\rho{}^b(x^\sigma(\lambda))} = \frac{4\pi i}{k}\delta^{ab}\epsilon_{\alpha\beta\gamma}\dot{x}^\beta(\lambda)\,\dot{x}^\gamma(\lambda)\,\langle U_\gamma(\gamma(\lambda),x_2)\,T_b T_a U_\gamma(x_1,\gamma(\lambda))\rangle$$

Since $\delta^{ab}T_aT_b$ is the Casimir and commutes with everything, by Schur's lemma $\delta^{ab}T_aT_b = c_2\mathbb{I}$. c_2 will depend on the representation, so we work in the fundamental representation. Here $c_2 = -\frac{1}{2}\left(N - \frac{1}{N}\right)$

$$\frac{\delta\langle U_\gamma(x_1,x_2)\rangle}{\delta x^\alpha(\lambda)} = \left\langle \frac{\delta U_\gamma(x_1,x_2)}{\delta x^\alpha(\lambda)} \right\rangle = \frac{4\pi i}{k}c_2\epsilon_{\alpha\beta\gamma}\dot{x}^\beta(\lambda)\,\dot{x}^\gamma(\lambda)\,\langle U_\gamma(x_1,x_2)\rangle$$

Variation of the Wilson Loop for a point change

For a closed loop, this becomes

$$\langle \delta W_K \rangle = \frac{4\pi i}{k} c_2 \nu \langle W_K \rangle$$

where

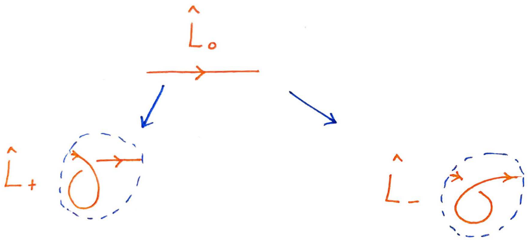
$$\nu = \epsilon_{\alpha\beta\gamma} \int_{S^1} \delta x^{\alpha}(\lambda)\, \dot{x}^{\beta}(\lambda)\, \dot{x}^{\gamma}(\lambda)\, \mathrm{d}\lambda$$

For a change happening at a single point $\delta x^{\alpha}(\lambda) = c^{\alpha} \delta(\lambda - \lambda^*)$ where λ^* is the point of change. In this case,

$$\nu = \begin{cases} +\frac{1}{2} & \text{out of the plane} \\ -\frac{1}{2} & \text{into the plane} \\ 0 & \text{in plane} \end{cases}$$

where the plane is $\mathrm{d}x^{\alpha}(\lambda^*) \wedge \mathrm{d}x^{\beta}(\lambda^*)$

Set of relations corresponding to R-move 1

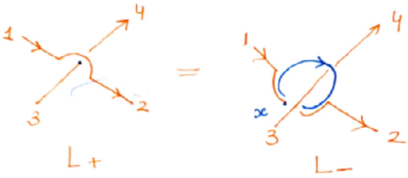


On appyling our formula for variation of a Wilson loop to these figures, we get

For $\alpha = 1 - \frac{2\pi i c_2}{k} + O\left(\frac{1}{k^2}\right)$

$$\left\langle W\left(\hat{L}_+\right)\right\rangle = \alpha \left\langle W\left(\hat{L}_0\right)\right\rangle$$
$$\left\langle W\left(\hat{L}_-\right)\right\rangle = \alpha^{-1} \left\langle W\left(\hat{L}_0\right)\right\rangle$$

Relations corresponding to R-moves 2 and 3



So L_+ and L are related by the insertion of a loop. Using a similar argument for two points, and the small loop expansion of $W = 1 - \epsilon^2 F$, we get

For $\beta = 1 + \frac{i\pi}{kN} + O\left(\frac{1}{k^2}\right)$ and $z = -\frac{2\pi i}{k} + O\left(\frac{1}{k^2}\right)$

$$\beta \langle W(L_+) \rangle - \beta^{-1} \langle W(L_-) \rangle = z \langle W(L_0) \rangle$$

Conclusion

Using the above two relations for $R1$, $R2$, $R3$, we showed that the polynomial defined as

$$P_K(\alpha, \beta, z) = \alpha^{-w(K)} \frac{\langle W(K) \rangle}{\langle W(O) \rangle}$$

satisfies

$$P_O = 1$$
$$tP_{L_+} - t^{-1} P_{L_-} = zP_{L_0}$$

This is the HOMFLY-PT polynomial introduced earlier. Here

$$t = \alpha\beta = 1 + \frac{i\pi N}{k} + O\left(\frac{1}{k^2}\right)$$
$$z = -\frac{2\pi i}{k} + O\left(\frac{1}{k^2}\right)$$

This matches with Witten's result for $N = 2$, the Jones Polynomial. His result was $t = e^{\frac{2\pi i}{k}}$.

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