SQFTs and Integral Modular Funtions*

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The main result of this report is that the partition function of a 2-dimensional minimally super symmetric quantum field theory is an integral modular function. The key idea is that these theories are not all conformal. Thus, it is a priori not expected that the partition function should be meromorphic, and $\mathrm{SL}_2(\mathbb{Z})$-invariant, with integer Fourier coefficients. This all comes out of super symmetry and evaluation of the partition function on the tori. Assuming that the partition function depends smoothly on the Riemann structure of the tori, having integral modular functions in the co-domain, makes the partition function a deformation invariant, i.e. all SQFTs that are path-connected (by RG flow or otherwise) and the SCFTs they flow to, map to the same integral modular function (IMF). We next compute what this form of partition function and supersymmetry means for the stress-energy tensor of the theory. We finally talk about the ring-homomorphism between the cobordism ring and ring of IMF, through super-symmetric sigma-models. This is introduced as the elliptic genus associated to the cobordism class of the target manifold.

INTRODUCTION

Wilson, Friedan and others in the 1970s conceptualised the 'Space of quantum field theories' or 'theory space'. Better understanding of this space is required for organising and classifying QFTs, allowing us to say when two theories are connected by finite variations of the couplings (also referred to as deformations) or by RG flows, i.e. when they are path connected. One of the main aims is to find bounds to the amount of information needed to uniquely specify a QFT, which will enabling us to estimate their number, that is the size of the 'theory space'. [Dou13] We wish to find *deformation invariants of QFTs*, i.e. quantities that do not change on perturbation, or going along RG flows. Thus, these quantities will be equal for a QFT and the CFT that it flows to under RG. and we know that CFTs can be thought of as landmarks in the space of QFTs. These invariants can help us understand the path-components of the space of QFTs using the CFT they flow to. [GJFW19] Reversely, we can try to quantify the space of QFTs geometrically and get a richer knowledge about the structure of the subspace containing CFTs. This well help in the classification of phases in condensed matter. [GJF19] if two CFTs have different invariants, the phases around them can not transition into each other.

We will work in the space of $\mathcal{N} = (0,1)$ supersymmetric QFTs as such deformation invariants are easier to find in this case and as it turns out are just the partition function evaluated on a torus.

In a boundary QFT, we evaluate our QFT not only on closed manifolds, but also on manifolds with boundary. The boundaries are assigned a Hilbert spaces while the manifold itself is assigned a linear map between these Hilbert spaces. In the supersymmetric case, our Hilbert spaces are $\mathbb{Z}_2$-graded (they can be written as $\mathcal{H}_0 \oplus \mathcal{H}_1$), direct sum of an odd and an even sector. We define the supersymmetric braiding of the tensor product of two super vector spaces V, W by

$$\begin{aligned}\tau\colon V\otimes W &\to W\otimes V\\ v\otimes w &\mapsto (-1)^{|v||w|}\, w\otimes v\end{aligned}$$

for elements of odd or even degree, and extend it by linearity.

We call an operator **even if it preserves the grading**, that is maps even setctor to even and odd to odd, an operator is **odd if it reverses the grading**. In block matrix form, $A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11}\end{pmatrix}$ with even parts on diagonal and odd parts off-diagonal. Any operator can be written as the sum of an even and odd operator. We can then define the super-commutator of operators on super vector spaces

$$[A,B] = AB - (-1)^{|A||B|}\, BA$$

for even ($|\,| = 0$) or odd ($|\,| = 1$) operators.This can be extended to all operators by linearity.
A super Lie algebra is a super vector space with a bilinear map $[-,-]: V\otimes V \to V$ such that

$$[A,B] = -\,(-1)^{|A||B|}\,[B,A]$$

for homogeneous elements of the super vector space. The bracket also satisfies a version of the Jacobi identity.

The above definition of the supercommutator of operators turns the space of operators into a *super Lie algebra*. The supertrace of any operator is defined as

$$\operatorname{str} A = \operatorname{tr} A_{00} - \operatorname{tr} A_{11}$$

where 00 and 11 components lie on the diagonal in the block matrix notation and tr is the normal trace. Also, (super)trace of a (super)commutator vanishes. This can be easily seen by taking cases. The superdimension is just the supertrace of identity. $\mathrm{sdim}A = \mathrm{dim}A_{00} - \mathrm{dim}A_{11}$

For $2D$ (Poincaré) supersymmetry we have super Lie algebra whose even part is the Poincaré algebra. It will have at least one of two super Poincaré algebras called the $\mathcal{N} = (0,1)$ and $\mathcal{N} = (1,0)$ algebras as its subalgebras, that is, the holomorphic sector (the "right movers") has no supersymmetry, while the anti-holomorphic sector (the "left movers") has $N = 1$ supersymmetry (or the other way around). We say that a 2 dimensional QFT has $\mathcal{N} = (0,1)$ supersymmetry when the fields defining the theory form a representation of the $\mathcal{N} = (0,1)$ super-Poincaré algebra.

This is the usual Poincaré algebra equipped with one extra element Q, satisfying

$$\{Q,Q\} = P_y - iP_x = -2i\bar{P}$$

and Q commutes with P_i, L

We wish to find a geometric space for the above supersymmetric algebra. We will call this $\mathbb{R}^{n|m}$. Superspace is an extension of ordinary spacetime to include extra anti-commuting coordinates. A complex super (cs) manifold of dimension $n|m$ is a space such that locally looks like $\mathbb{R}^{n|m}$, the $\mathbb{C}$-valued scalar fields on it look like

$$C^{\infty}\left(\mathbb{R}^{n},\mathbb{C}\right)\otimes\Lambda\left[\theta_1,\ldots,\theta_m\right]$$

where $\Lambda\left[\theta_1,\ldots,\theta_m\right]$ is the algebra over $\mathbb{C}$ generated by m elements satisfying $\theta_i\theta_j=-\theta_j\theta_i$.

Define the space $\mathbb{R}^{2|1}$ to be parameterised by three coordinates $(z,\bar{z},\theta)$ such that $z,\bar{z}$ are complex and θ is Grassmann. This is a cs manifolds of dimension $2|1$. We give it a super Lie group structure by defining multiplication by

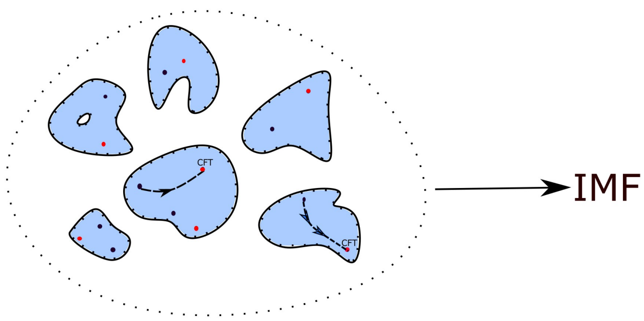
$$\left(z_1,\bar{z}_1,\theta_1\right)+_s\left(z_2,\bar{z}_2,\theta_2\right)=\left(z_1+z_2,\bar{z}_1+\bar{z}_2+\theta_1\theta_2,\theta_1+\theta_2\right) \tag{1}$$

This is the exponentiation of the $\mathcal{N}=(0,1)$ super Poincaré algebra and hence we see the interaction of $\bar{z}$ and θ

RESULT: DEFORMATION INVARIANT OF 2D SQFT

The partition function of a 2d $\mathcal{N}=(0,1)$ supersymmetric QFT evaluated on a torus is an integral modular function. Further, this map is surjective. This is a deformation invariant [ST11]. We will only prove the first part of the theorem.

An integral modular function is a meromorphic function $f\colon\mathbb{H}\to\mathbb{C}$ invariant under the $\mathrm{SL}_2(\mathbb{Z})$ action and has integral Fourier coefficients.



Construction and Proof

$C_{l,f}$ is a super-cylinder i.e. its two boundaries are S_l (super circle with circumference l. The cylinder is quotient of $\mathbb{R}^{2|1}$ by $l\mathbb{Z}$. Similarly, a torus $T_{l,f}$ is a quotient by $l\left(\mathbb{Z}f+\mathbb{Z}1\right)$. Next we consider a few geometric facts,

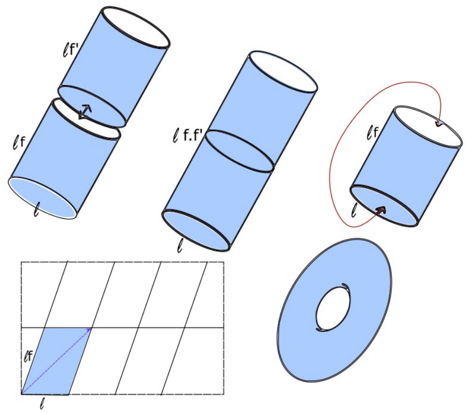
$$C_{l,f}\circ C_{l,f'}=C_{l,f+_s f'} \tag{2}$$

Gluing two cylinders along S_l gives another cylinder with parameter $l,f+_s f'$ and, gluing the two boundaries of the cylinder, we get a torus,

$$\widehat{C_{l,f}}=T_{l,f} \tag{3}$$

If we take $f\to f+_s(1,1,0)$ the lattice remains the same and hence it just corresponds to taking different fundamental domains of the same lattice (also preserving the boundary circle S_l) of the cylinder, hence

$$C_{l,f+_s(1,1,0)}=C_{l,f} \tag{4}$$



And, finally if we take a real torus (no supersymmetric direction, $f=(\tau,\bar{\tau},0)$), i.e., take $T_{l,(\tau,\bar{\tau},0)}$, we have

$$T_{g\cdot(l,\tau)}=T_{l,\tau} \tag{5}$$

where $g\in\mathrm{SL}_2(\mathbb{Z})$ acts on l,τ as

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot(l,\tau)=\left(l\left|c\tau+d\right|,\frac{a\tau+b}{c\tau+d}\right)$$

keeps the lattice unchanged. For any $T_{l,f}$, we

$$Z\left[T_{l,f}\right]=Z\left[\widehat{C_{l,f}}\right]=\mathrm{str}\;Z\left[C_{l,f}\right]$$

. We get a super-trace because we take periodic boundary conditions for bosonic and anti-periodic boundary conditions for the fermionic part of the operator. Since $f\in\mathbb{R}^{2|1}$, we can write $f=(z,\bar{z},\theta)$ for $z\in\mathbb{H}$. Thus we have

$$Z\left[C_{l,f}\right]=A_l\left(z,\bar{z}\right)+\theta B_l\left(z,\bar{z}\right) \tag{6}$$

where A_l,B_l are operators on $Z\left[S_l\right]$. Z is an even quantity as it is observable and thus since θ is odd, B must be odd too. For the same reasons A must be even.

The relation 4 gives us that

$$\begin{aligned}A_l\left(z+1,\bar{z}+1\right)=&A_l\left(z,\bar{z}\right)\\B_l\left(z+1,\bar{z}+1\right)=&B_l\left(z,\bar{z}\right)\end{aligned}$$

And, similarly, from 1 and 2 we obtain the following equation on taking the partition function Z,

$$\begin{aligned}&\left(A_l\left(z_1,\bar{z}_1\right)+\theta_1B_l\left(z_1,\bar{z}_1\right)\right)\left(A_l\left(z_2,\bar{z}_2\right)+\theta_2B_l\left(z_2,\bar{z}_2\right)\right)\\=&A_l\left(z_1+z_2,\bar{z}_1+\bar{z}_2+\theta_1\theta_2\right)\\&+\left(\theta_1+\theta_2\right)B_l\left(z_1+z_2,\bar{z}_1+\bar{z}_2+\theta_1\theta_2\right)\end{aligned}$$

On Taylor expanding the right hand side we get, and identifing like terms, we get the relations

$$\begin{aligned}A_l\left(z_1,\bar{z}_1\right)A_l\left(z_2,\bar{z}_2\right)=&A_l\left(z_1+z_2,\bar{z}_1+\bar{z}_2\right)\\A_l\left(z_1,\bar{z}_1\right)B_l\left(z_2,\bar{z}_2\right)=&B_l\left(z_1,\bar{z}_1\right)A_l\left(z_2,\bar{z}_2\right)=B_l\left(z_1+z_2,\bar{z}_1+\bar{z}_2\right)\\\bar{\partial}A\left(z_1+z_2,\bar{z}_1+\bar{z}_2\right)=&-B\left(z_1,\bar{z}_1\right)B\left(z_2,\bar{z}_2\right)\end{aligned} \tag{7}$$

Consider a torus with no extension in the super directions, that is, a torus with $f=(z,\bar{z},0)$. Then we define a function

$$\mathcal{Z}\colon\mathbb{R}^{+}\times\mathbb{H}\to\mathbb{C}$$

defined by $\mathcal{Z}\left(l,z,\bar{z}\right)=Z\left[T_{l,(z,\bar{z},0)}\right]$. Taking $\theta=0$ in 6

$$\mathcal{Z}\left(l,z,\bar{z}\right)=\mathrm{str}\;A_l\left(z,\bar{z}\right)$$

From the last relation in 7, we find that

$$\bar{\partial}A_l(z,\bar{z}) = -B_l\left(\frac{z}{2},\frac{\bar{z}}{2}\right)^2 = -\frac{1}{2}\left\{B_l\left(\frac{z}{2},\frac{\bar{z}}{2}\right), B_l\left(\frac{z}{2},\frac{\bar{z}}{2}\right)\right\}$$

Next, we take the super-trace of the above and also use the fact that str commutes with $\bar{\partial}$

$$\bar{\partial}\,\mathrm{str}\,A_l(z,\bar{z}) = 0 \tag{8}$$

We use the fact that str commutes with $\bar{\partial}$ and the supertrace of supercommutator vanishes. This shows that $A_l(z,\bar{z})$ is meromorphic.

From Eq 7, $A_l(z_1,\bar{z}_1)A_l(z_2,\bar{z}_2) = A_l(z_1+z_2,\bar{z}_1+\bar{z}_2)$, we see that all the $A_l(z,\bar{z})$ commute (for fixed l). Thus, we diagonalise all the $A_l(z,\bar{z})$ simultaneously. This will give us eigenvalues $\lambda_i^{(l)}(z,\bar{z}) \in \mathbb{C}$. Then,

$$\begin{aligned}
&A_l(z_1,\bar{z}_1)A_l(z_2,\bar{z}_2) = A_l(z_1+z_2,\bar{z}_1+\bar{z}_2)\\
&\implies \lambda_i^{(l)}(z_1,\bar{z}_1)\lambda_i^{(l)}(z_2,\bar{z}_2) = \lambda_i^{(l)}(z_1+z_2,\bar{z}_1+\bar{z}_2)\\
&\implies \lambda_i^{(l)}(z_1,\bar{z}_1) = 0 \text{ or } \lambda_i^{(l)}(z,\bar{z}) = \exp\left(2\pi i\left(a_l z - b_l\bar{z}\right)\right)
\end{aligned}$$

Since $A_l(z+1,\bar{z}+1) = A_l(z,\bar{z})$, we obtain that $a_l - b_l \in \mathbb{Z}$.

Let $H_{a,b}(z,\bar{z},l)$ be the eigenspace of $A_l(z,\bar{z})$ corresponding to the eigenvalue $q^a\bar{q}^b$ where $q = e^{2\pi i z}$. It will only consist of 0 if that particular pairing of a,b does not give rise to an eigenvalue of $A_l(z,\bar{z})$. Let $H_0(z,\bar{z},l)$ be the eigenspace corresponding to the eigenvalue 0.

The superdimensions of $H_{a,b}(z,\bar{z},l), H_0(z,\bar{z},l)$ are integers and thus, when we vary $z,\bar{z},l$ continuously, they cannot change continuously (as integers are a discrete space). Thus, these quantities are independent of $z,\bar{z},l$. We denote these constant values by $h_{a,b}, h_0$

Now, expressing the supertrace as eigenvalues times the superdimension of the eigenvalue space, where sdim is

$$\begin{aligned}
\mathrm{str}\,A_l(z,\bar{z}) =& \sum_{a,b} q^a\bar{q}^b \mathrm{sdim}\,H_{a,b}(z,\bar{z},l)\\
=& \sum_{a,b} q^a\bar{q}^b h_{a,b}
\end{aligned}$$

Since str $A_l(z,\bar{z})$ is meromorphic from 8 there is no $\bar{z}$ dependence, we obtain that $h_{a,b} = 0$ for $b \neq 0$. Thus

$$\mathrm{str}\,A_l(z,\bar{z}) = \sum_{a\in\mathbb{Z}} q^a h_{a,0}$$

Two above two things together show that $\mathcal{Z}(l,z,\bar{z})$ is meromorphic, independent of l and has a integral Fourier coefficients. Since $\mathcal{Z}$ is meromorphic, it can only have a pole as $z \to i\infty$ $(q \to 0)$. Since meromorphic functions can't have essential singularities, the sum over a needs to be bounded below.

Finally to show modular invariance of the partition function on modular transformation on z, given any $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, we act on l,z as

$$l, z \mapsto l\left|cz+d\right|, \frac{az+b}{cz+d}$$

Any two lattices linked by an elements of $\mathrm{SL}_2(\mathbb{Z})$ give rise to the same torus, thus we have

$$\mathcal{Z}\left(l,\frac{az+b}{cz+d}\right) = \mathcal{Z}\left(l\left|cz+d\right|,\frac{az+b}{cz+d}\right) = \mathcal{Z}(l,z)$$

First equality is because of the fact that $\mathcal{Z}$ does not depend on l so we change it, and the second one as the values of l,τ are linked by $\mathrm{SL}_2(\mathbb{Z})$, and hence are actually the same torus, so the partition function must take the same value.

This proves the result that partition function on torus is an integral modular function. A consequence of the discreteness of integers and *continuity of the partition function* with respect to the Riemannian structure on the torus, implies that any two SQFTs connected by a path must give rise to the same modular form. We actually have a surjection

$$\pi_0\left(SQFTs\right) \to \text{Integral modular functions}$$

where π_0 denotes the connected components of the space.Any SQFT is path connected to an SCFT by RG flow, and thus we see that the map below is also surjective.

$$\text{SCFTs} \to \text{Integral modular functions}$$

Although we theoretically know that all modular functions come from a CFT, not many examples of $\mathcal{N}(0,1)$ SCFTs are know, The few which are known are from string theory and next, we discuss supersymmteric sigma models.

AN EQUIVALENT FORMULATION IN TERMS OF THE STRESS-ENERGY TENSOR

For any QFT, we have

$$\begin{aligned}
\frac{1}{Z[M,g]}\frac{\delta Z[M,g]}{\delta g^{\mu\nu}(x)} =& \frac{1}{Z[M,g]}\frac{\delta}{\delta g^{\mu\nu}(x)}\int \mathcal{D}\phi \exp\left(iS_{M,g}[\phi]\right)\\
=& \frac{1}{Z[M,g]}\int \mathcal{D}\phi\exp\left(iS_{M,g}[\phi]\right) i\frac{\delta S_{M,g}[\phi]}{\delta g^{\mu\nu}(x)}\\
=& i\left\langle T_{\mu\nu}(x)\right\rangle\sqrt{g(x)}
\end{aligned}$$

The torus $T_{l,\tau}$ is defined as the quotient of $\mathbb{R}^2$ by the lattice $l(\tau\mathbb{Z}+\mathbb{Z}1)$ with metric inherited from $\mathbb{R}^2$. Let the coordinates on $\mathbb{R}^2$ be $z,\bar{z}$. We define coordinates a,b as

$$z = l(a+b\tau)$$

for $0 \leq a,b < 1$. In these coordinates, the metric is

$$g_{\mu\nu}(a,b) = l^2\begin{pmatrix} 1 & \mathrm{Re}(\tau)\\ \mathrm{Re}(\tau) & |\tau|^2\end{pmatrix}$$

Thus

$$g^{\mu\nu}(a,b) = \frac{1}{l^2\,\mathrm{Im}(\tau)^2}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}(\tau)\\ -\mathrm{Re}(\tau) & 1\end{pmatrix}$$

We want to see how our space changes when we vary $l,\tau,\bar{\tau}$ (which is why we chose the a,b coordinates, so that all changes are in the metric)

$$\begin{aligned}
\frac{\partial g^{\mu\nu}(a,b)}{\partial l} =& \frac{-2}{l^3\,\mathrm{Im}(\tau)^2}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}(\tau)\\ -\mathrm{Re}(\tau) & 1\end{pmatrix}\\
\frac{\partial g^{\mu\nu}(a,b)}{\partial\bar{\tau}} =& \frac{-i}{l^2\,\mathrm{Im}(\tau)^3}\begin{pmatrix} \tau\,\mathrm{Re}(\tau) & \frac{i}{2}\mathrm{Im}(\tau)-\tau\\ \frac{i}{2}\mathrm{Im}(\tau)-\tau & 1\end{pmatrix}\\
\frac{\partial g^{\mu\nu}(a,b)}{\partial\tau} =& \frac{i}{l^2\,\mathrm{Im}(\tau)^3}\begin{pmatrix} \bar{\tau}\,\mathrm{Re}(\tau) & -\frac{i}{2}\mathrm{Im}(\tau)-\bar{\tau}\\ -\frac{i}{2}\mathrm{Im}(\tau)-\bar{\tau} & 1\end{pmatrix}
\end{aligned}$$

For a $2d$ QFT, let us calculate the variation of $Z[T_{l,\tau}]$ with $p = l,\tau,\bar{\tau}$.

$$\frac{1}{Z[T_{l,\tau}]}\frac{\partial Z[T_{l,\tau}]}{\partial p} = i\int_{[0,1]^2} \mathrm{d}a\mathrm{d}b\sqrt{g(a,b)}\left\langle T_{\mu\nu}(a,b)\right\rangle\frac{\partial}{\partial p}g^{\mu\nu}(a,b)$$

So now we calculate the expression within the integral.

$$\begin{aligned}
T_{\mu\nu}(a,b)\frac{\partial}{\partial l}g^{\mu\nu}(a,b) =& \frac{-2}{l^3\operatorname{Im}(\tau)^2}\big(T_{aa}(a,b)|\tau|^2 \\
&- 2\operatorname{Re}(\tau)T_{ab}(a,b) + T_{bb}(a,b)\big) \\
=& -\frac{8}{l}T_{z\bar{z}}(z,\bar{z}) \\
T_{\mu\nu}(a,b)\frac{\partial}{\partial\bar{\tau}}g^{\mu\nu}(a,b) =& \frac{-i}{l^2\operatorname{Im}(\tau)^3}\big(T_{aa}(a,b)\tau\operatorname{Re}(\tau) \\
&- 2\left(\frac{i}{2}\operatorname{Im}(\tau)-\tau\right)T_{ab}(a,b) \\
&+ T_{bb}(a,b)\big) \\
=& \frac{2i}{\operatorname{Im}(\tau)}\left(T_{\bar{z}\bar{z}}(z,\bar{z}) - T_{z\bar{z}}(z,\bar{z})\right) \\
T_{\mu\nu}(a,b)\frac{\partial}{\partial\tau}g^{\mu\nu}(a,b) =& \frac{i}{l^2\operatorname{Im}(\tau)^3}\big(T_{aa}(a,b)\bar{\tau}\operatorname{Re}(\tau) \\
&- 2\left(-\frac{i}{2}\operatorname{Im}(\tau)-\bar{\tau}\right)T_{ab}(a,b) \\
&+ T_{bb}(a,b)\big) \\
=& \frac{2i}{\operatorname{Im}(\tau)}\left(T_{z\bar{z}}(z,\bar{z}) - T_{zz}(z,\bar{z})\right)
\end{aligned}$$

In deriving these, we used the transformation laws of the energy momentum tensor between the (a,b) coordinates and $(z,\bar{z})$ coordinates, given by

$$\begin{pmatrix} T_{aa}(a,b) \\ T_{ab}(a,b) \\ T_{bb}(a,b) \end{pmatrix} = l^2 \begin{pmatrix} 1 & 2 & 1 \\ \tau & 2\operatorname{Re}(\tau) & \bar{\tau} \\ \tau^2 & 2|\tau|^2 & \bar{\tau}^2 \end{pmatrix} \begin{pmatrix} T_{zz}(z,\bar{z}) \\ T_{z\bar{z}}(z,\bar{z}) \\ T_{\bar{z}\bar{z}}(z,\bar{z}) \end{pmatrix}$$

Now, we know that $\mathrm{d}a\mathrm{d}a\sqrt{g(a,b)}$ is invariant under isometries, thus

$$\mathrm{d}a\mathrm{d}b\sqrt{g(a,b)} = \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g(z,\bar{z})} = \frac{1}{2}\mathrm{d}z\mathrm{d}\bar{z}$$

Given any function $f\colon M \to \mathbb{C}$, define

$$f^{\text{avg}} = \frac{1}{\operatorname{Vol}_g(M)}\int_M f(x)\,d\operatorname{Vol}_g$$

Thus, we end up with the expressions

$$\begin{aligned}
\frac{1}{Z[T_{l,\tau}]}\frac{\partial Z[T_{l,\tau}]}{\partial l} =& -8il\operatorname{Im}(\tau)\langle T^{\text{avg}}_{z\bar{z}}\rangle \\
\frac{1}{Z[T_{l,\tau}]}\frac{\partial Z[T_{l,\tau}]}{\partial\bar{\tau}} =& 2l^2\langle T^{\text{avg}}_{z\bar{z}} - T^{\text{avg}}_{\bar{z}\bar{z}}\rangle \\
\frac{1}{Z[T_{l,\tau}]}\frac{\partial Z[T_{l,\tau}]}{\partial\tau} =& 2l^2\langle T^{\text{avg}}_{zz} - T^{\text{avg}}_{z\bar{z}}\rangle
\end{aligned}$$

where we have used the fact that $\operatorname{Area}(T_{l,\tau}) = l^2\operatorname{Im}(\tau)$.

For $\mathcal{N}=(0,1)$ supersymmetric QFTs, we have proved that there is no $l,\bar{\tau}$ dependence. We see that this is equivalent to the fact that

$$\begin{aligned}
\langle T^{\text{avg}}_{z\bar{z}}\rangle =& \langle T^{\text{avg}}_{\bar{z}\bar{z}}\rangle = 0 \\
\langle T^{\text{avg}}_{zz}\rangle =& \frac{t(\tau)}{2l^2}
\end{aligned}$$

Further, from the fact that $Z[l,\tau]$ is an integral modular form, we know that t is the logarithmic derivative of an integral modular function. These three relations are an equivalent way of saying that the partition function is an integral modular function with no $l,\bar{\tau}$ dependence.

Since $T^{\text{avg}}_{\mu\nu}$ is the 0^{th} mode of it's Fourier expansion, and is thus related to the L_{-2} operators (as the stress-energy tensor has weight 2), the above relations can be expressed in terms of the vacuum expectation of the L_{-2} operators.

2D $\mathcal{N}=(0,1)$ LINEAR SIGMA MODELS

A sigma model is a collection of QFTs (one for each manifold X) such that the fields on a closed manifold M are maps $M \to X$. Here, X is called the target manifold. The partition function is given by

$$Z_X[M] = \int_{\operatorname{Maps}[M,X]} \mathcal{D}\psi\exp(iS[\psi]) \tag{9}$$

We take $M = T_{l,\tau}$. Then we obtain a map, which maps the target manifold X to the partition function of the sigma-model evaluated on a torus.

$$\begin{aligned}
\Phi\colon \text{Manifolds} \to& \text{Functions mapping } l,\tau \text{ to } \mathbb{C} \\
X \mapsto& Z_X[T_{l,\tau}]
\end{aligned}$$

Firstly, we look at what happens on a manifolds $X = X_1 \amalg X_2$, that is taking disjoint union. If M is connected, smooth maps into X will either map all of M into X_1 or into X_2. So we have

$$\operatorname{Maps}[M,X] \cong \operatorname{Maps}[M,X_1] \amalg \operatorname{Maps}[M,X_2]$$

Thus,

$$\begin{aligned}
&Z_{X_1\amalg X_2}[M] = \int_{\operatorname{Maps}[M,X]} \mathcal{D}\psi\exp(iS[\psi]) \\
&= \int_{\operatorname{Maps}[M,X_1]} \mathcal{D}\psi\exp(iS[\psi]) + \int_{\operatorname{Maps}[M,X_1]} \mathcal{D}\psi\exp(iS[\psi]) \\
&= Z_{X_1}[M] + Z_{X_2}[M]
\end{aligned}$$

Since $T_{l,\tau}$ is connected, this implies that

$$\Phi[X_1 \amalg X_2] = \Phi[X_1] + \Phi[X_2] \tag{10}$$

Next, consider $X = X_1 \times X_2$. Then

$$\operatorname{Maps}[M,X] \cong \operatorname{Maps}[M,X_1] \times \operatorname{Maps}[M,X_2]$$

Given a $\psi\colon M \to X$, let ψ_1,ψ_2 be the projections on X_1, X_2. These completely determine ψ. A sigma model is called linear if

$$S_{X_1\times X_2}[\psi_1,\psi_2] = S_{X_1}[\psi_1] + S_{X_2}[\psi_2]$$

For linear sigma models, we have

$$\begin{aligned}
&Z_{X_1\times X_2}[M] \\
&= \int_{\operatorname{Maps}[M,X_1]\times\operatorname{Maps}[M,X_2]} \mathcal{D}\psi_1\mathcal{D}\psi_2\exp(iS[\psi_1]+iS[\psi_2]) \\
&= \int_{\operatorname{Maps}[M,X_1]} \mathcal{D}\psi_1\exp(iS[\psi_1]) \times \int_{\operatorname{Maps}[M,X_2]} \mathcal{D}\psi_2\exp(iS[\psi_2]) \\
&= Z_{X_1}[M]\,Z_{X_2}[M]
\end{aligned}$$

Thus, taking $M = T_{l,\tau}$

$$\Phi(X_1\times X_2) = \Phi(X_1)\,\Phi(X_2) \tag{11}$$

After observing the action of Φ on disjoint union and Cartesian product of target manifolds, we look at the empty set and single point. On taking $X = \emptyset$, if M is nonempty, then $\mathrm{Maps}\,[M, \emptyset] = \emptyset$. Thus, the partition function is going to be an integral over an empty set, and thus 0. Thus

$$\Phi\,(\emptyset) = 0 \tag{12}$$

Finally, $X = \mathrm{pt}$, then $\mathrm{Maps}\,[M, \mathrm{pt}] \cong \mathrm{pt} = \psi_{\mathrm{pt}}$, that is there is only one field configuration, mapping everything to a single point. In a partition function $\mathrm{D}\Phi$ is supposed to represent the measure on the space of maps, but the measure on a discrete space is just the cardinality. Now, consider $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$ in the target manifold . From linearity of the sigma model, we obtain $S_{\mathrm{pt}}\,[\psi_{\mathrm{pt}}] = S_{\mathrm{pt}}\,[\psi_{\mathrm{pt}}] + S_{\mathrm{pt}}\,[\psi_{\mathrm{pt}}]$

Thus, $S\,[\psi_{\mathrm{pt}}] = 0$. The partition function becomes

$$Z_{\mathrm{pt}}\,[M] = \int_{\mathrm{pt}} \mathrm{dcard}\exp\,(0) = \mathrm{card}\,(\mathrm{pt}) = 1$$

$$\Phi\,(\mathrm{pt}) = 1 \tag{13}$$

Given two manifolds X_0, X_1 of dimension d, a cobordism between them Σ is a $d+1$ dimensional manifold such that $\partial\Sigma = \bar{X}_1 \amalg X_2$. We can slice Σ into fibres X_t for each $t \in [0, 1]$ such that X_t match the boundaries at $t = 0, 1$. Now, consider a linear supersymmetric sigma model, then we obtain a map

$$\begin{aligned} p_\Sigma\colon\ [0, 1] &\to \text{Integral Modular Functions} \\ t &\mapsto Z_{X_t}\,[T_{l,\tau}] \end{aligned}$$

Since integral modular forms are discrete (as integer coeffecients cannot deform), we know that p_Σ must be a constant function. Thus, if there a cobordism from X_1, X_2, then

$$\Phi\,(X_1) = \Phi\,(X_2) \tag{14}$$

We say two manifolds are cobordism equivalent if there is a

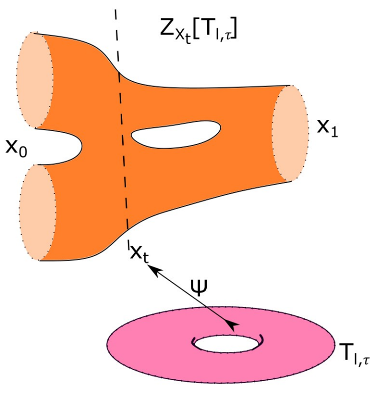


FIG. 1.

cobordism between them. Define the cobordism ring as

$$\Omega_n = \frac{\text{set of manifolds of dimension } n}{\text{cobordism equivalence}}$$

$$\Omega = \bigoplus_{n=0}^{\infty} \Omega_n$$

Addition is given by disjoint union and multiplication by the cartesian product.

From Eq. 10, 11, 12, 13 and 14, we see that the ma Φ is a ring homomorphism

$$\Phi\colon \Omega \to \text{Integral modular functions}$$

We showed that given any 2 dimensional $\mathcal{N} = (0, 1)$ supersymmetric linear sigma model Z, we obtain an elliptic genus defined by

$$X \mapsto Z_X\,[T_{l,\tau}]$$

In [Och87], a universal elliptic genus was constructed mathematically such that post-composition by evaluation on any torus produces the corresponding elliptic genus. Witten [Wit87] showed that this universal elliptic genus comes from the large volume limit of the partition function of $E_8 \times E_8$ heterotic string theory in the manner described above. We can see that since we do not have an l dependence, taking the limit of large volume does not change anything. This is called the *Witten genus*.

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