Supersymmetric QFTs and Integral Modular Functions

Based on [ST11]

Nivedita, Indian Institute of Technology, Kanpur.

Motivation

- We wish to find deformation invariants of QFTs, i.e. quantities that do not change on perturbation, or going along RG flows.
- These quantities can help us understand the path-components of the space of QFTs. [GJFW19]
- This is similar to how we want to classify phases in condensed matter. [GJF19]
- We will work in the space of $\mathcal{N} = (\mathbf{0}, \mathbf{1})$ supersymmetric QFTs as such deformation invariants are easier to find in this case.

Super vectorspaces

Symmetric monoidal category of complex supervector spaces:

- Objects: $\mathbb{Z}_2$-graded complex vector spaces $V_0 \oplus V_1$
- Morphisms: Grading preserving maps
- Monoidal structure: The tensor product is given by $(V_0 \oplus V_1) \otimes (W_0 \oplus W_1) = (V_0 \otimes W_0 \oplus V_1 \otimes W_1) \oplus (V_0 \otimes W_1 \oplus V_1 \otimes W_0)$
- Unit: $\mathbb{C} \oplus \mathbf{0}$
- The left and right unitors and the associator are inherited from the category of vector spaces
- Braiding: $\tau \colon V \otimes W \to W \otimes V$ is given by

$$v \otimes w \to (-1)^{|v||w|}\, w \otimes v$$

for homogeneous $v \in V, w \in W$

A super Lie algebra is a super vectorspace with a map $[-,-] : \boldsymbol{V} \otimes \boldsymbol{V} \to \boldsymbol{V}$ such that

$$[-,-] = -[-,-] \circ \boldsymbol{\tau}$$

and also satisfies a Jacobi identity.
Super trace is defined as the map that makes the diagram commute

$$\begin{tikzcd}
\boldsymbol{V} \otimes \boldsymbol{V}^* \arrow[r] \arrow[d] & \mathbf{Hom}(\boldsymbol{V}, \boldsymbol{V}) \arrow[d, "\mathbf{str}"] \\
\boldsymbol{V}^* \otimes \boldsymbol{V} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

and $\mathrm{sdim}\, \boldsymbol{V} = \mathrm{str}\, \mathbf{id}_{\boldsymbol{V}}$
We will actually want to work with locally convex vector spaces, but these will involve more analytic aspects.

Supersymmetry

By (Poincaré) supersymmetry we mean a super Lie algebra whose even part is the Poincare algebra. Minimally we can have one of two super Poincare algebras called the $\mathcal{N} = (0, 1)$ and $\mathcal{N} = (1, 0)$ algebras.

We say that a 2 dimensional QFT has $\mathcal{N} = (0, 1)$ supersymmetry when the fields defining the theory form a representation of the $\mathcal{N} = (0, 1)$ super-Poincaré algebra.
This is the usual Poincaré algebra equipped with an extra element Q, satisfying

$$\{Q, Q\} = -2i\bar{P}$$

and Q commutes with $P, \bar{P}, L$
We wish to find a geometric space corresponding the above supersymmetric algebra. We will call this $\mathbb{R}^{n|m}$.

cs manifolds

A complex super (cs) manifold of dimension $n|m$ is a Haussdorf second-countable space M_{red} with a sheaf of graded$\mathbb{C}$-algebras $\mathcal{O}$ such that around every point of M_{red} there is an open set such that

$$(U, \mathcal{O}|_U) \simeq \left(\mathbb{R}^n, \mathcal{O}^{n|m}\right)$$

where $\mathcal{O}^{n|m}$ is a sheaf on $\mathbb{R}^n$ such that for any open set $V \subseteq \mathbb{R}^n$

$$\mathcal{O}^{n|m}(V) = C^{\infty}(V, \mathbb{C}) \otimes \Lambda[\theta_1, \ldots, \theta_m]$$

where $\Lambda[\theta_1, \ldots, \theta_m]$ the exterior algebra over $\mathbb{C}$ with m generators.
We define a map of cs manifolds to be a map of ringed spaces, and just as in normal manifolds.

- Define the space $\mathbb{R}^{2|1}$ to be parametrised by three cooridnates $(z, \bar{z}, \theta)$ such that $z, \bar{z}$ are complex and θ is Grassmann.
- This is a cs manifold of dimension $2|1$. We give it a super Lie group structure by defining multiplication by

$$(z_1, \bar{z}_1, \theta_1) +_s (z_2, \bar{z}_2, \theta_2) = (z_1 + z_2, \bar{z}_1 + \bar{z}_2 + \theta_1\theta_2, \theta_1 + \theta_2)$$

  This is the exponentiation of the $\mathcal{N} = (0, 1)$ super Poincare algebra.

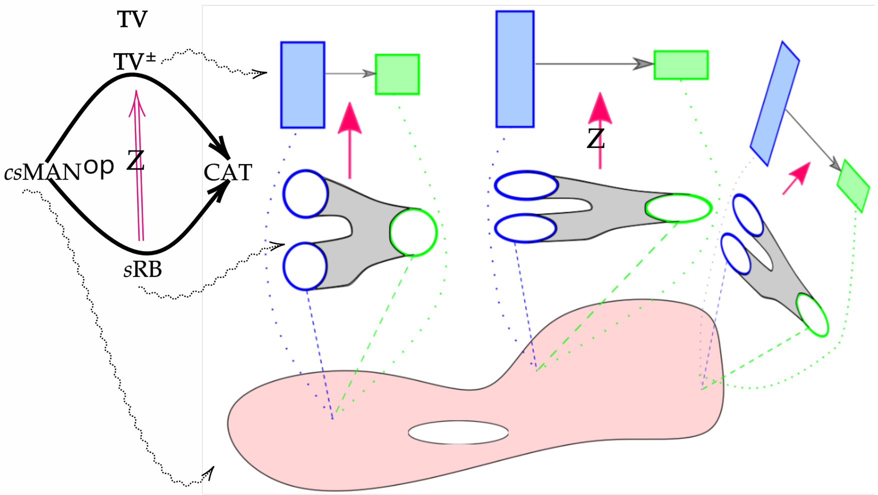
Some definitions

A smooth category is a functor $\mathbf{csMan^{op}} \to \mathbf{Cat}$. We define the smooth categories $\mathbf{sRB}, \mathbf{TV}^{\pm}$ as

- The category $\mathbf{sRB}_{\boldsymbol{S}}$: objects are smooth (quasi) bundles of cs-manifolds $\boldsymbol{U} \to \boldsymbol{S}$ with fibres of dimension $\boldsymbol{d}|\mathbf{1}$ which are equipped with a fiberwise super Riemannian structure (fiberwise here means that the local vector fields D, B are vertical); A morphism is an embedding of cs-manifolds (cobordism) that is a bundle map, and preserves the fiberwise super Riemannian structure.
- Objects of $\mathbf{TV}_{\boldsymbol{S}}$ are super vector bundles over $\boldsymbol{S}$, defined sheaf theoretically (graded modules) to accomodate the super-structure of $\boldsymbol{S}$

We define a SQFT to be a smooth functor from $\mathbf{sRB} \to \mathbf{TV_S}^{\pm}$

Background



Aim

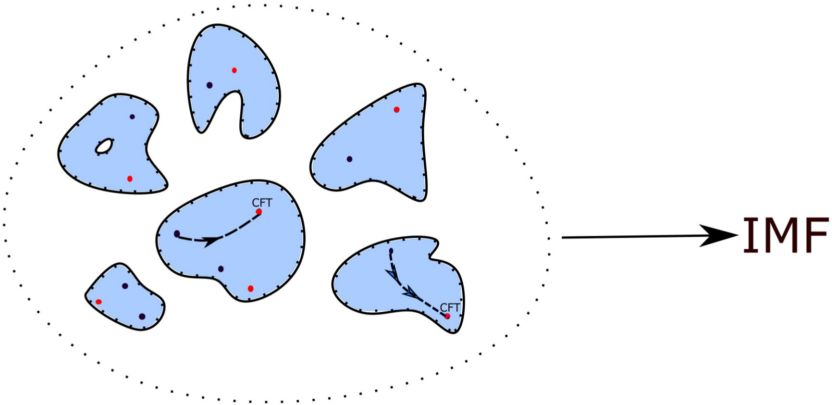
Main theorem [ST11]

The partition function of a 2d $\mathcal{N} = (\mathbf{0}, \mathbf{1})$ supersymmetric QFT evaluated on a torus is an integral modular function.

We will attempt to prove this theorem.

- An integral modular function is a meromorphic function $\boldsymbol{f} : \mathbb{H} \to \mathbb{C}$ invariant under the $\mathbf{SL_2}(\mathbb{Z})$ action and has integral Fourier coefficients

Result



Theorem 1

Some Geometric Facts

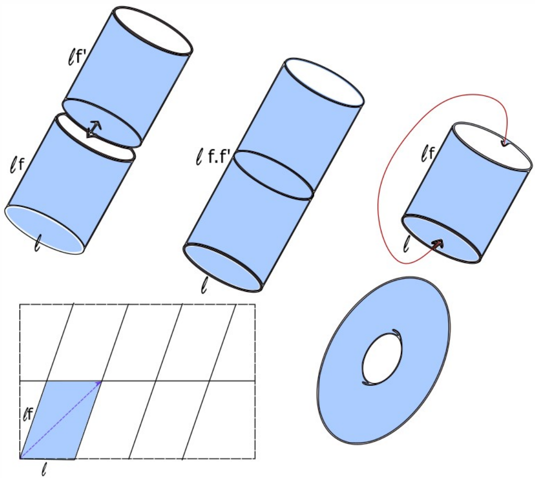


Figure: $\boldsymbol{C_{l,f}}$ *is a super-cylinder i.e. its two boundaries are* $\boldsymbol{S_l}$ *(super circle with circumference* $\boldsymbol{l}$*. The cylinder is quotient of* $\mathbb{R}^{\mathbf{2|1}}$ *by a lattice, with parameters* $\boldsymbol{l}$ *and* $\boldsymbol{lf}$

Theorem 1

Geometric facts

- $C_{l,f} \circ C_{l,f'} = C_{l,f+_s f'}$ Gluing two cylinders along S_l gives another cylinder with parameter $f +_s f'$
- $\widehat{C_{l,f}} = T_{l,f}$ gluing the two boundaries of the cylinder, we get a torus
- $C_{l,f+_s(1,1,0)} = C_{l,f}$ if we take $f \to f +_s (1,1,0)$
- Given a real torus (no supersymmetric direction), i.e., $T_{l,(\tau,\bar{\tau},0)}$, we have

$$T_{g\cdot(l,\tau)} = T_{l,\tau}$$

where $g \in \mathrm{SL}_2(\mathbb{Z})$ acts on l, τ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (l, \tau) = \left(l\,|c\tau + d|\,, \frac{a\tau + b}{c\tau + d} \right)$$

Theorem 1

Form of partition function of of a torus

For any $\boldsymbol{T_{l,f}}$, we

$$\boldsymbol{Z}\left[\boldsymbol{T_{l,f}}\right] = \boldsymbol{Z}\left[\widehat{\boldsymbol{C_{l,f}}}\right] = \text{str}\ \boldsymbol{Z}\left[\boldsymbol{C_{l,f}}\right]$$

Since $\boldsymbol{f} \in \mathbb{R}^{\mathbf{2|1}}$, we can write $\boldsymbol{f} = (\boldsymbol{z}, \bar{\boldsymbol{z}}, \theta)$ for $\boldsymbol{z} \in \mathbb{H}$. Thus we have

$$\boldsymbol{Z}\left[\boldsymbol{C_{l,f}}\right] = \boldsymbol{A_l}\left(\boldsymbol{z}, \bar{\boldsymbol{z}}\right) + \theta \boldsymbol{B_l}\left(\boldsymbol{z}, \bar{\boldsymbol{z}}\right)$$

where $\boldsymbol{A_l}, \theta \boldsymbol{B_l}$ are operators on $\boldsymbol{Z}\left[\boldsymbol{S_l}\right]$. $\boldsymbol{Z}$ is an even quantity as it is observable and thus since θ is odd, $\boldsymbol{B}$ must be odd too. For the same reasons $\boldsymbol{A}$ must be even.
The relation $\boldsymbol{C}_{\boldsymbol{l},(z+1,\bar{z}+1,\theta)} = \boldsymbol{C}_{\boldsymbol{l},(z,\bar{z},\theta)}$ gives us that

$$\begin{aligned}
\boldsymbol{A_l}\left(\boldsymbol{z}+\mathbf{1}, \bar{\boldsymbol{z}}+\mathbf{1}\right) =& \boldsymbol{A_l}\left(\boldsymbol{z}, \bar{\boldsymbol{z}}\right) \\
\boldsymbol{B_l}\left(\boldsymbol{z}+\mathbf{1}, \bar{\boldsymbol{z}}+\mathbf{1}\right) =& \boldsymbol{B_l}\left(\boldsymbol{z}, \bar{\boldsymbol{z}}\right)
\end{aligned}$$

Theorem 1

Additional Relations

Since $\boldsymbol{C}_{\boldsymbol{I},z_1,\bar{z}_1,\theta_1} \circ \boldsymbol{C}_{\boldsymbol{I},z_2,\bar{z}_2,\theta_2} = \boldsymbol{C}_{\boldsymbol{I},z_1+z_2,\bar{z}_1+\bar{z}_2+\theta_1\theta_2,\theta_1+\theta_2}$, we obtain the following equatiom by acting by $\boldsymbol{Z}$

$$\begin{aligned}
&\left(\boldsymbol{A}_{\boldsymbol{I}}\left(z_1,\bar{z}_1\right)+\theta_1\boldsymbol{B}_{\boldsymbol{I}}\left(z_1,\bar{z}_1\right)\right)\left(\boldsymbol{A}_{\boldsymbol{I}}\left(z_2,\bar{z}_2\right)+\theta_2\boldsymbol{B}_{\boldsymbol{I}}\left(z_2,\bar{z}_2\right)\right)\\
=&\boldsymbol{A}_{\boldsymbol{I}}\left(z_1+z_2,\bar{z}_1+\bar{z}_2+\theta_1\theta_2\right)\\
&+\left(\theta_1+\theta_2\right)\boldsymbol{B}_{\boldsymbol{I}}\left(z_1+z_2,\bar{z}_1+\bar{z}_2+\theta_1\theta_2\right)
\end{aligned}$$

On taylor explaning and identifing like terms, we get the relations

$$\begin{aligned}
\boldsymbol{A}_{\boldsymbol{I}}\left(z_1,\bar{z}_1\right)\boldsymbol{A}_{\boldsymbol{I}}\left(z_2,\bar{z}_2\right)=&\boldsymbol{A}_{\boldsymbol{I}}\left(z_1+z_2,\bar{z}_1+\bar{z}_2\right)\\
\boldsymbol{A}_{\boldsymbol{I}}\left(z_1,\bar{z}_1\right)\boldsymbol{B}_{\boldsymbol{I}}\left(z_2,\bar{z}_2\right)=&\boldsymbol{B}_{\boldsymbol{I}}\left(z_1,\bar{z}_1\right)\boldsymbol{A}_{\boldsymbol{I}}\left(z_2,\bar{z}_2\right)=\boldsymbol{B}_{\boldsymbol{I}}\left(z_1+z_2,\bar{z}_1+\bar{z}_2\right)\\
\bar{\partial}\boldsymbol{A}\left(z_1+z_2,\bar{z}_1+\bar{z}_2\right)=&-\boldsymbol{B}\left(z_1,\bar{z}_1\right)\boldsymbol{B}\left(z_2,\bar{z}_2\right)
\end{aligned} \tag{1}$$

Theorem 1

Suppressing the 'super' dimension

Consider a torus with no extension in the super directions, that is, a torus with $\boldsymbol{f} = (\boldsymbol{z}, \bar{\boldsymbol{z}}, \mathbf{0})$. Then we define a function

$$\mathcal{Z} : \mathbb{R}^+ \times \mathbb{H} \to \mathbb{C}$$

defined by $\mathcal{Z}(\boldsymbol{l}, \boldsymbol{z}, \bar{\boldsymbol{z}}) = \boldsymbol{Z}\left[\boldsymbol{T}_{\boldsymbol{l},(\boldsymbol{z},\bar{\boldsymbol{z}},\mathbf{0})}\right]$ Then we have

$$\mathcal{Z}(\boldsymbol{l}, \boldsymbol{z}, \bar{\boldsymbol{z}}) = \mathrm{str}\ \boldsymbol{A}_{\boldsymbol{l}}(\boldsymbol{z}, \bar{\boldsymbol{z}})$$

Theorem 1

Meromorphic

$$\bar{\partial} \boldsymbol{A_l}(z,\bar{z}) = -\boldsymbol{B_l}\left(\frac{z}{2},\frac{\bar{z}}{2}\right)^2 = -\frac{1}{2}\left\{\boldsymbol{B_l}\left(\frac{z}{2},\frac{\bar{z}}{2}\right),\boldsymbol{B_l}\left(\frac{z}{2},\frac{\bar{z}}{2}\right)\right\}$$

$$\bar{\partial}\ \mathrm{str}\ \boldsymbol{A_l}(z,\bar{z}) = 0$$

We use the fact that **str** commutes with $\bar{\partial}$ and the supertrace of supercommutator vanishes.

Expansion

Since $\boldsymbol{A_l}\left(z_1, \bar{z}_1\right) \boldsymbol{A_l}\left(z_2, \bar{z}_2\right) = \boldsymbol{A_l}\left(z_1 + z_2, \bar{z}_1 + \bar{z}_2\right)$, we see that all the $\boldsymbol{A_l}\left(z, \bar{z}\right)$ commute (for fixed $\boldsymbol{l}$). Thus, we diagonalise all the $\boldsymbol{A_l}\left(z, \bar{z}\right)$ simultaeneously. This will give us eigenvalues $\lambda_i^{(l)}\left(z, \bar{z}\right) \in \mathbb{C}$. Since,

$$\begin{aligned}
&\boldsymbol{A_l}\left(z_1, \bar{z}_1\right) \boldsymbol{A_l}\left(z_2, \bar{z}_2\right) = \boldsymbol{A_l}\left(z_1 + z_2, \bar{z}_1 + \bar{z}_2\right) \\
&\quad \implies \lambda_i^{(l)}\left(z_1, \bar{z}_1\right) \lambda_i^{(l)}\left(z_2, \bar{z}_2\right) = \lambda_i^{(l)}\left(z_1 + z_2, \bar{z}_1 + \bar{z}_2\right) \\
&\quad \implies \lambda_i^{(l)}\left(z_1, \bar{z}_1\right) = 0 \text{ or } \lambda_i^{(l)}\left(z, \bar{z}\right) = \exp\left(2\pi i\left(a_l z - b_l \bar{z}\right)\right)
\end{aligned}$$

Since $\boldsymbol{A_l}\left(z + 1, \bar{z} + 1\right) = \boldsymbol{A_l}\left(z, \bar{z}\right)$, we obtain that $a_l - b_l \in \mathbb{Z}$.

Theorem 1

Let $H_{a,b}(z, \bar{z}, l)$ be the eigenspace of $A_l(z, \bar{z})$ corresponding to the eigenvalue $q^a \bar{q}^b$ where $q = e^{2\pi i z}$. It will only consist of 0 if that particular pairing of a, b does not give rise to an eigenvalue of $A_l(z, \bar{z})$. Let $H_0(z, \bar{z}, l)$ be the eigenspace corresponding to the eigenvalue $\mathbf{0}$.
The superdimensions of $H_{a,b}(z, \bar{z}, l), H_0(z, \bar{z}, l)$ are integers and thus, when we vary $z, \bar{z}, l$ continuously, they cannot change (as integers are a discrete space). Thus, these quantities are independent of $z, \bar{z}, l$. We denote these constant values by $h_{a,b}, h_0$
Now,

$$\begin{aligned}
\text{str } A_l(z, \bar{z}) &= \sum_{a,b} q^a \bar{q}^b \text{sdim } H_{a,b}(z, \bar{z}, l) \\
&= \sum_{a,b} q^a \bar{q}^b h_{a,b}
\end{aligned}$$

Since $\mathbf{str}\ \boldsymbol{A_I}\left(\boldsymbol{z}, \bar{\boldsymbol{z}}\right)$ is meromorphic, we obtain that $\boldsymbol{h_{a,b}} = \mathbf{0}$ for $\boldsymbol{b} \neq \mathbf{0}$. Thus

$$\mathbf{str}\ \boldsymbol{A_I}\left(\boldsymbol{z}, \bar{\boldsymbol{z}}\right) = \sum_{\boldsymbol{a} \in \mathbb{Z}} \boldsymbol{q}^{\boldsymbol{a}} \boldsymbol{h_{a,0}}$$

Two above two things together show that $\mathcal{Z}\left(\boldsymbol{I}, \boldsymbol{z}, \bar{\boldsymbol{z}}\right)$ is meromorphic, independent of $\boldsymbol{I}$ and has a integral fourier coefficients.

Since $\mathbf{str}\ \boldsymbol{A_I}$ exists, and as $|\boldsymbol{q}| < \mathbf{1}$ we have $|\boldsymbol{q}|^{-\mathbf{1}} > \mathbf{1}$, the sum must terminate below.

Theorem 1
Modular invariance

Given any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we act on l, τ as

$$l, \tau \mapsto l \left| c\tau + d \right|, \frac{a\tau + b}{c\tau + d}$$

Any two lattices linked by an elements of $\mathrm{SL}_2\left(\mathrm{Z}\right)$ give rise to the same torus, thus we have

$$\mathcal{Z}\left(l, \frac{a\tau + b}{c\tau + d}\right) = \mathcal{Z}\left(l \left| c\tau + d \right|, \frac{a\tau + b}{c\tau + d}\right) = \mathcal{Z}\left(l, \tau\right)$$

First equality is because of the fact that $\mathcal{Z}$ does not depend on l, and the second one as the values of l, τ are linked by $\mathrm{SL}_2\left(\mathbb{Z}\right)$, and hence are actually the same torus, so the partition function must take the same value.

Theorem 1
Deformation invariant

A consequence of the discreteness of integers and continuity of the partition function with respect to the Riemannian structure on the torus, implies that any two SQFTs connected by a path must give rise to the same modular form. We actually have a surjection

$$\pi_0\left(\boldsymbol{SQFTs}\right) \to \text{Integral modular functions}$$

Any SQFT is path connected to an SCFT by RG flow, and thus we see that the map below is also surjective.

$$\boldsymbol{SCFTs} \to \text{Integral modular functions}$$

Sigma Models

A sigma model is a collection of QFTs (one for each manifold X) such that the fields on a closed manifold M are maps $M \to X$. That is, the partition function is given by

$$Z_X[M] = \int_{\psi \in \mathbf{Maps}[M,X]} \mathcal{D}\psi \exp(iS[\psi])$$

Now, take $M = T_{I,\tau}$. Then we obtain a map

$$\begin{aligned} \Phi \colon \mathbf{Manifolds} &\to \text{Functions mapping } I, \tau \text{ to } \mathbb{C} \\ X &\mapsto Z_X[T_{I,\tau}] \end{aligned}$$

Respects $\amalg$

Now, we look at what happens on a manifolds $X = X_1 \amalg X_2$. If M is connected, smooth maps into X will either map all of M into X_1 or into X_2. So we have

$$\mathrm{Maps}\left[M, X\right] \cong \mathrm{Maps}\left[M, X_1\right] \amalg \mathrm{Maps}\left[M, X_2\right]$$

Thus,

$$Z_{X_1 \amalg X_2}\left[M\right] = \int_{\psi \in \mathrm{Maps}[M,X]} \mathcal{D}\psi \exp\left(iS\left[\psi\right]\right)$$

$$= \int_{\psi \in \mathrm{Maps}[M,X_1]} \mathcal{D}\psi \exp\left(iS\left[\psi\right]\right) + \int_{\psi \in \mathrm{Maps}[M,X_1]} \mathcal{D}\psi \exp\left(iS\left[\psi\right]\right)$$

$$= Z_{X_1}\left[M\right] + Z_{X_2}\left[M\right]$$

Since $T_{I,\tau}$ is connected, this implies that

$$\Phi\left[X_1 \amalg X_2\right] = \Phi\left[X_1\right] + \Phi\left[X_2\right]$$

Respects $\times$

Consider $X = X_1 \times X_2$. Then

$$\mathrm{Maps}\,[M, X] \cong \mathrm{Maps}\,[M, X_1] \times \mathrm{Maps}\,[M, X_2]$$

Given a $\psi\colon M \to X$, let ψ_1, ψ_2 be the projections on X_1, X_2. These completely determine ψ. A sigma model is called linear if

$$S_{X_1 \times X_2}[\psi_1, \psi_2] = S_{X_1}[\psi_1] + S_{X_2}[\psi_2]$$

For linear sigma models, we have

$$\begin{aligned} Z_{X_1 \times X_2}[M] &= \int_{\mathrm{Maps}[M,X_1] \times \mathrm{Maps}[M,X_2]} \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp\left(iS[\psi_1] + iS[\psi_2]\right) \\ &= \int_{\mathrm{Maps}[M,X_1]} \mathcal{D}\psi_1 \exp\left(iS[\psi_1]\right) \\ &\quad \times \int_{\mathrm{Maps}[M,X_2]} \mathcal{D}\psi_2 \exp\left(iS[\psi_2]\right) \\ &= Z_{X_1}[M]\, Z_{X_2}[M] \end{aligned}$$

Thus, taking $\boldsymbol{M} = \boldsymbol{T}_{\boldsymbol{I},\boldsymbol{\tau}}$

$$\Phi\left(\boldsymbol{X}_1 \times \boldsymbol{X}_2\right) = \Phi\left(\boldsymbol{X}_1\right)\Phi\left(\boldsymbol{X}_2\right)$$

Repects additive identity

Take $\boldsymbol{X} = \emptyset$. If $\boldsymbol{M}$ is nonempty, then $\mathbf{Maps}\left[\boldsymbol{M}, \emptyset\right] = \emptyset$. Thus, the partition function is going to be an integral over an empty set, and thus $\mathbf{0}$. Thus

$$\Phi\left(\emptyset\right) = \mathbf{0}$$

Respects multiplicative identity

Finally, $\boldsymbol{X} = \mathbf{pt}$, then $\mathbf{Maps}\,[\boldsymbol{M}, \mathbf{pt}] \cong \mathbf{pt} = \psi_{\mathbf{pt}}$, that is there is only one field configuration, mapping everything to a single point. In a partition function $\mathbf{D\Phi}$ is supposed to represent the measure on the space of maps, but the measure on a discrete space is just the cardinality. Now, consider $\mathbf{pt} \times \mathbf{pt} \cong \mathbf{pt}$ in the target manifold . From linearity of the sigma model, we obtain
$\boldsymbol{S}_{\mathbf{pt}}\left[\psi_{\mathbf{pt}}\right] = \boldsymbol{S}_{\mathbf{pt}}\left[\psi_{\mathbf{pt}}\right] + \boldsymbol{S}_{\mathbf{pt}}\left[\psi_{\mathbf{pt}}\right]$
Thus, $\boldsymbol{S}\left[\psi_{\mathbf{pt}}\right] = \mathbf{0}$. The partition function becomes

$$\boldsymbol{Z}_{\mathbf{pt}}\,[\boldsymbol{M}] = \int_{\mathbf{pt}} \mathbf{dcard} \exp\left(\mathbf{0}\right) = \mathbf{card}\left(\mathbf{pt}\right) = \mathbf{1}$$

$$\mathbf{\Phi}\left(\mathbf{pt}\right) = \mathbf{1} \tag{2}$$

Respects the cobordism equivalence

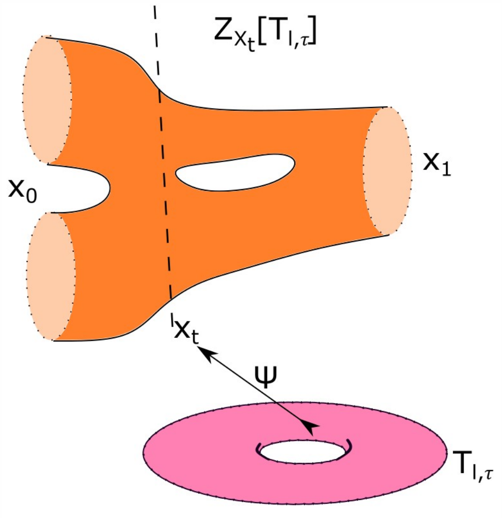
Given two manifolds X_0, X_1, a cobordism between them Σ is a manifold such that $\partial\Sigma = \bar{X}_1 \amalg X_2$. Slice Σ into fibres X_t for each $t \in [0, 1]$ such that X_t match the boundaries at $t = 0, 1$. Now, consider a supersymmetric sigma model, then we obtain a map

$$\begin{aligned} p_\Sigma \colon\ [0, 1] &\to \text{Integral Modular Functions} \\ t &\mapsto Z_{X_t}\left[T_{I,\tau}\right] \end{aligned}$$

Since integral modular forms are discrete (as integer coeffecients cannot deform), we know that p_Σ must be a constant function. Thus, if there a cobordism from X_1, X_2, then

$$\Phi\left(X_1\right) = \Phi\left(X_2\right)$$

Sigma models



Figure

Genera

We say two manifolds are cobordism equivalent if there is a cobordism between them. Define the cobordism ring as

$$\Omega_n = \frac{\text{set of manifolds of dimension } n}{\text{cobordism equivalence}}$$

$$\Omega = \bigoplus_{n=0}^{\infty} \Omega_n$$

Addition is given by disjoint union and multiplication by the cartesian product.
An genus is a ring homomorphism

$$\Phi \colon \Omega \to \text{Integral modular functions}$$

In the previous slides, we just showed that given any 2 dimensional $\mathcal{N} = (\mathbf{0}, \mathbf{1})$ supersymmetric linear sigma model $\boldsymbol{Z}$, we obtain an genus defined by

$$\boldsymbol{X} \mapsto \boldsymbol{Z}_{\boldsymbol{X}}\left[\boldsymbol{T}_{\boldsymbol{I},\tau}\right]$$

References I

Davide Gaiotto and Theo Johnson-Freyd, *Symmetry protected topological phases and generalized cohomology*, Journal of High Energy Physics **2019** (2019), no. 7, 10.1007/JHEP05(2019)007 (English), arXiv:1712.07950.

Davide Gaiotto, Theo Johnson-Freyd, and Edward Witten, *A note on some minimally supersymmetric models in two dimensions*, arXiv:1902.10249, March 2019.

References II

- Stephan Stolz and Peter Teichner, *Supersymmetric field theories and generalized cohomology*, Mathematical Foundations of Quantum Field Theory and Perturbative String Theory (Hisham Sati and Urs Schreiber, eds.), vol. 83, Proceedings of Symposia in Pure Mathematics, no. 83, American Mathematical Society, Providence, Rhode Island, 1st ed., December 2011, arXiv:1108.0189, pp. 279–340 (English).

Questions

More on cs- manifolds

- We also note that any cs manifold induces a real manifold structure on M_{red}
- Vector fields: A homogeneous vector field X of degree $|X| = 0, 1$ is a graded derivation on $\mathcal{O}_M(M)$. i.e., it is a linear map such that

$$X(fg) = X(f)\,g + (-1)^{|X||f|}\,fX(g)$$

for homogeneous elements $f, g \in \mathcal{O}_M(M)$. A general vector field is the sum of a homogeneous vector field of degree 0 and one of of degree 1.

Super Riemannian structure on $2|1$ cs manifolds

Define a super Riemannian structure to be a

- An open cover U_i
- Two vector fields D_i, B_i on each U_i such that
  - B_{red} is the complex conjugate of D^2_{red} and these two are linearly independent at each point of M_{red}
- The restrictions of D_i, B_i and D_j, B_j to $U_i \cap U_j$ are related by
  - There is a smooth function $f: M \to \mathrm{U}(1)$ such that

$$D_j|_{U_i \cap U_j} = \bar{f} D_i|_{U_i \cap U_j} \qquad B_j|_{U_i \cap U_j} = f^2 B_i|_{U_i \cap U_j}$$

We say two such structures are equivalent if their union is another such structure. We call a super Riemannian structure flat if $[D_i, B_i] = 0$

Motivation

The above definitions are not simple generalisations of the usual definitions of a metric and a flat metric. They are justified by the following theorems:

A super Riemannian structure on a cs manifold $\boldsymbol{M}$ of dimension $\mathbf{2|1}$ induces a metric as well as a spin structure on $\boldsymbol{M}_{\mathbf{red}}$.

The metric induced on $\boldsymbol{M}_{\mathbf{red}}$ by a flat super Riemannian structure on $\boldsymbol{M}$ is flat.

Super Riemannian structure on $\mathbb{R}^{2|1}$

- Take
$$\boldsymbol{D} = \partial_\theta - \theta\partial_{\bar{z}} \qquad \boldsymbol{B} = -\partial_{\bar{z}}$$
It is easy to see that these satisfy the axioms and the y commute, and thus give us a flat super Riemannian structure on $\mathbb{R}^{2|1}$
- The group structure we defined on $\mathbb{R}^{2|1}$ preserves this structure. This is what allows us the define the cylinders $\boldsymbol{C_{l,f}} \in \mathsf{sRB}\left(\boldsymbol{S_l}, \boldsymbol{S_l}\right)$ as the cobordisms
$$\mathbb{R}^{2|1}/\mathbb{Z}\boldsymbol{l} \xrightarrow{\text{id}} \mathbb{R}^{2|1}/\mathbb{Z}\boldsymbol{l} \xleftarrow{\text{addition by } \boldsymbol{lf}} \mathbb{R}^{2|1}/\mathbb{Z}\boldsymbol{l}$$
where we defined $\boldsymbol{S_l}$ as the tuple
$$\left(\mathbb{R}^{2|1}/\mathbb{Z}\boldsymbol{l}, \mathbb{R}^{1|1}/\mathbb{Z}\boldsymbol{l}, \mathbb{R}^{2|1}_+/\mathbb{Z}\boldsymbol{l}, \mathbb{R}^{2|1}_-/\mathbb{Z}\boldsymbol{l}\right)$$

We define the category $\mathbf{sRB}_{S}$ (for a cs-manifold of arbitrary dimension) as

- Objects: Quasibundles over S such that the fibres are $2|1$ dimensional cs-manifolds with super Riemannian structure (vertically). For each fibre U, we have a quadruple (U, Y, U^+, U^-) such that
  - Y is a $1|1$ cs manifold
  - $U^\pm$ are open such that $U^+ \cup U^- = U \setminus Y$
  - Y which is in the closure of both $U^\pm$
- Morphisms: Quasibundles over S such that the fibres Σ are $2|1$ dimensional cs-manifolds with Riemannian structure such that on each fibre there are open subsets $V_i \subseteq U_i$ which contain Y_i such that
  - There are isometries $\iota_i \colon V_i \to \Sigma$ such that $\iota_1\left(V_1^+ \cup Y_1\right)$ and $\iota_2\left(V_2^- \cup Y_2\right)$ are disjoint
  - $\Sigma \setminus \left(\iota_1\left(V_1^+\right) \cup \iota_2\left(V_2^-\right)\right)$ is compact

$\mathsf{TV}^{\pm}$

We define the category $\mathsf{TV}^{\pm}_{\boldsymbol{S}}$ (for a cs-manifold of arbitrary dimension) as

- Objects: Triples $(\boldsymbol{V}^{+}, \boldsymbol{V}^{-}, \boldsymbol{\mu})$ where $\boldsymbol{V}^{\pm}$ are quasibundles over $\boldsymbol{S}$ such that the fibres super vector spaces and $\boldsymbol{\mu}\colon \boldsymbol{V}^{-} \otimes \boldsymbol{V}^{+} \to \mathbb{C}_{\boldsymbol{S}}$ ($\mathbb{C}_{\boldsymbol{S}}$ is the trivial bundle over $\boldsymbol{S}$ with fibre $\mathbb{C} \oplus \mathbf{0}$
- A morphism from $(\boldsymbol{V}^{+}, \boldsymbol{V}^{-}, \boldsymbol{\mu}_{\boldsymbol{V}})$ to $(\boldsymbol{W}^{+}, \boldsymbol{W}^{-}, \boldsymbol{\mu}_{\boldsymbol{W}})$ is a pair of grading preserving vector bundle maps $(\boldsymbol{T}^{+}\colon \boldsymbol{V}^{+} \to \boldsymbol{W}^{+}, \boldsymbol{T}^{-}\colon \boldsymbol{W}^{-} \to \boldsymbol{V}^{-})$ such that $\boldsymbol{\mu}_{\boldsymbol{V}}\left(\boldsymbol{T}^{-}\boldsymbol{w}^{-}, \boldsymbol{v}^{+}\right) = \boldsymbol{\mu}_{\boldsymbol{W}}\left(\boldsymbol{w}^{-}, \boldsymbol{T}^{+}\boldsymbol{v}^{+}\right)$ where $\boldsymbol{v}^{+} \in \boldsymbol{V}^{+}$ and $\boldsymbol{w}^{-} \in \boldsymbol{W}^{-}$

This category has the nice property that whenever a morphism $\boldsymbol{T}\colon \boldsymbol{V} \to \boldsymbol{W}$ factorises as

$$\boldsymbol{V} \simeq \mathbb{C} \otimes \boldsymbol{V} \xrightarrow{\boldsymbol{T}_2 \otimes \mathrm{id}_{\boldsymbol{V}}} \boldsymbol{W} \otimes \boldsymbol{U} \otimes \boldsymbol{V} \xrightarrow{\boldsymbol{T}_1 \otimes \boldsymbol{T}_1} \boldsymbol{W} \otimes \boldsymbol{C} \simeq \boldsymbol{W}$$

then $\boldsymbol{T}^+$ is nuclear. Since every cobordism factors this way, the image under the QFT will factor this way, and this shows that the image of a cobordism under a QFT will always be nuclear.