# Constructing a $3,2$-TQFT with defects from a $3,2,1$-TQFT

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## 1 Introduction

Let $\mathcal{Z}\colon \mathrm{Cob}^{\mathrm{or}}_{3,2,1} \to \mathrm{LinCat}$ be a 3-2-1 TQFT and $\mathcal{C} = \mathcal{Z}\left(S^1\right)$. Given closed 2-manifold with marked points we construct a vector spaces and given a cobordism between them with an embedded ribbon graph, we construct a linear map.

Let $\tilde{\Sigma}$ is a marked surface. Each marked point is labelled by a sign $+$or $-$, and has a germ of arcs $[\gamma] : (-\delta, \delta) \to \tilde{\Sigma}$. In order to consider $\tilde{\Sigma}$ as an object in the category $\mathrm{Cob}_{3,2}\left(\mathcal{C}\right)$, where $\mathcal{C}$ is a modular tensor catgory, each marked point is additionally coloured by an object of $\mathcal{C}$. Let $\tilde{\Sigma}$ have $k_+$ (resp. $k_-$) points marked by $+$ (resp. $-$) with labels $A_1, \ldots, A_{k_+}$ (resp. $B_1, \ldots, B_{k_-}$).

For our construction, we first choose embeddings $F_i\colon D^2 \hookrightarrow \tilde{\Sigma}$ such that $F_i\left(0\right) = p_i$ and the germ of $(-1,1) \hookrightarrow D^2 \xrightarrow{F_i} \tilde{\Sigma}$ at 0 matches the germ $[\gamma]$ at $p_i$, and none of the images of $F_i$ intersect. Define $\Sigma$ to be $\tilde{\Sigma}$ minus the interiors of the embedded disks. We have a 3-manifold $\tilde{M}$ such that $\partial\tilde{M} = \tilde{\Sigma}$ with embedded ribbon graph. Each ribbon is a path or a string $r_j$ with a framing $t_j$, $r_j\colon\ [0,1] \hookrightarrow \tilde{M}$ such that the two end points of the ribbon are either the marked points on the boundary $\tilde{\Sigma}$ (marked $+$ if ribbon starts there and $-$ if it ends). A ribbon can also end on a coupon. The framing is $t_j\colon\ [0,1] \to \mathrm{T}\tilde{M}$ which is a lift of $r_j$ across the projection $\mathrm{T}\tilde{M} \to \tilde{M}$. We choose the embeddings $\Phi\colon D^2 \times [0,1] \hookrightarrow \tilde{M}$ around the ribbons such that $\Phi|_{D^2\times\{0\}} = F_i$ and $\Phi|_{D^2\times\{1\}} = F'_i$ such that $i, i'$ are two marked points or points on a coupon. Further, $\Phi|_{\{0\}\times[0,1]} = r_j$ and $\partial_x\Phi|_{\{(0,0)\}\times[0,1]} = t_j$. Around each coupon we choose an embedding $C\colon\ [0,1]\times[-1,1]\times[0,1] \hookrightarrow \tilde{M}$ such that $C|_{y=0}$ is the coupon and the end points of the tubular neighbourhoods are fully covered by $C_{\mathrm{z}=0}$ and $C_{\mathrm{z}=1}$. We define $M$ to be $\tilde{M}$ minus the interiors of the embeddings $\Phi, C$.

## 2 Evaluating $\tilde{\mathcal{Z}}$ on on objects

Define

$$\tilde{\mathcal{Z}}\left(\tilde{\Sigma}\right) = \mathrm{Hom}_{\mathcal{C}^{\boxtimes k_-}}\left(B_1 \boxtimes \cdots \boxtimes B_{k_-}, \mathcal{Z}\left(\Sigma\right)\left(A_1 \boxtimes \cdots \boxtimes A_{k_+}\right)\right)$$

**Lemma 1.** *We have canonical isomorphisms* $\tilde{\mathcal{Z}}\left(\tilde{\emptyset}\right) \cong \mathbb{C}$ *and* $\tilde{\mathcal{Z}}\left(\tilde{\Sigma}_1 \sqcup \tilde{\Sigma}_2\right) \cong \tilde{\mathcal{Z}}\left(\tilde{\Sigma}_1\right) \otimes_{\mathbb{C}} \tilde{\mathcal{Z}}\left(\tilde{\Sigma}_2\right)$

**Lemma 2.** *Given a marked surface $\tilde{\Sigma}$, define $\tilde{\Sigma}^{+}$ to have the same underlying manifold but all the $-$-marked points are changed to $+$ and their labels are replaced by their (right) duals. Then $\tilde{\mathcal{Z}}\left(\tilde{\Sigma}\right)$ is canonically isomorphic to $\tilde{\mathcal{Z}}\left(\tilde{\Sigma}^{+}\right)$.*

*Proof.* To prove this, consider a surface with genus $g$ and $k_{+}$ points marked with $+$ and labelled by $A_1, \ldots A_{k_+}$ and $k_-$ points marked with $-$ and labelled by $B_1, \ldots B_{k_-}$. First assume $k_+ \geq 1$ We work inductively, suppose $k_- \geq 2$. We consider the same manifold labelled by $k_+ + 1$ points marked $+$ with labels $A_1 \ldots, A_{k_+}, B_{k_-}^{\vee}$ and $k_- - 1$ points marked $-$ with labels $B_1 \ldots, B_{k_- -1}$. First we show these have the same vector space. We want a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{C}^{\boxtimes k_-}}\left(B_1 \boxtimes \cdots \boxtimes B_{k_-}, \mathcal{Z}\left(\Sigma^{g,k_+,k_-}\right)\left(A_1 \boxtimes \cdots \boxtimes A_{k_+}\right)\right) \cong$$
$$\mathrm{Hom}_{\mathcal{C}^{\boxtimes (k_- -1)}}\left(B_1 \boxtimes \cdots \boxtimes B_{k_- -1}, \mathcal{Z}\left(\Sigma^{g,k_+ +1,k_- -1}\right)\left(A_1 \boxtimes \cdots \boxtimes A_{k_+} \boxtimes B_{k_-}^{\vee}\right)\right)$$

Note $\mathcal{Z}\left(\Sigma^{g,k_+,k_-}\right) \cong \mathcal{Z}\left(\Sigma^{0,1,k_-}\right) \circ \left(\mathcal{Z}\left(\Sigma^{0,2,1}\right) \circ \mathcal{Z}\left(\Sigma^{0,1,2}\right)\right)^{\circ g} \circ \mathcal{Z}\left(\Sigma^{0,k_+,1}\right)$ and $Z\left(\Sigma^{0,2,1}\right) : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ is the tensor product and $Z\left(\Sigma^{0,1,2}\right) : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$ is the comultiplication functor $\Delta$ given by

$$\Delta\left(X\right) = \bigoplus_{i \in \mathcal{I}} X \otimes U_i \boxtimes U_i^{\vee}$$

So we have $\mathcal{Z}\left(\Sigma^{0,2,1}\right) \circ \mathcal{Z}\left(\Sigma^{0,1,2}\right)(X) \cong X \otimes D$ where $D = \bigoplus_{i \in \mathcal{I}} U_i \otimes U_i^{\vee}$. Thus, this hom-set equivalence becomes an equivalence of the type

$$\mathrm{Hom}_{\mathcal{C}^{\boxtimes k_-}}\left(B_1 \boxtimes \cdots \boxtimes B_{k_-}, \mathcal{Z}\left(\Sigma^{0,1,k_-}\right)\left(A_1 \otimes \cdots \otimes A_{k_+} \otimes D^{\otimes g}\right)\right) \cong$$
$$\mathrm{Hom}_{\mathcal{C}^{\boxtimes (k_- -1)}}\left(B_1 \boxtimes \cdots \boxtimes B_{k_- -1}, \mathcal{Z}\left(\Sigma^{0,1,k_- -1}\right)\left(A_1 \otimes \cdots \otimes A_{k_+} \otimes B_{k_-}^{\vee} \otimes D^{\otimes g}\right)\right)$$

Now, we know $\Sigma^{0,k,1}$ is the left adjoint to $\Sigma^{0,1,k}$ for $k \geq 1$ and $\Sigma^{0,k,1}$ tensors its arguments together. Since $k_- \geq 2$, we need an equivalence between

$$\mathrm{Hom}_{\mathcal{C}}\left(B_1 \otimes \cdots \otimes B_{k_-}, A_1 \otimes \cdots \otimes A_{k_+} \otimes D^{\otimes g}\right) \cong$$
$$\mathrm{Hom}_{\mathcal{C}}\left(B_1 \otimes \cdots \otimes B_{k_- -1}, A_1 \otimes \cdots \otimes A_{k_+} \otimes B_{k_-}^{\vee} \otimes D^{\otimes g}\right)$$

Such an isomorphism exists canonically. Thus by induction we can always reduce to the case where there is single $B$ on the left side of the hom. We can then just move that over to the right side as a dual.

If $k_+ = 0$, a similar argument works but uses the adjunction of $\Sigma^{0,1,0}$ and $\Sigma^{0,0,1}$ also. □

# 3 Evaluating $\tilde{\mathcal{Z}}$ on morphisms

**Lemma 3.** *For any cobordism $\tilde{M} \colon \tilde{\Sigma}_{\mathrm{in}} \to \tilde{\Sigma}_{\mathrm{out}}$ there exist cobordisms $\tilde{M}_i \colon \tilde{\Sigma}_i \to \tilde{\Sigma}_{i+1}$ $(i = 1, \ldots, n)$ such that $\tilde{M} = \tilde{M}_{n-1} \circ \cdots \circ \tilde{M}_1$ and each of the $M_i$ contain at most a single coupon and each ribbon in $M_i$ is connected to the boundary.*

Starting with $\tilde{M}$ as a cobordism between $\tilde{\Sigma}_{\text{in}} \to \tilde{\Sigma}_{\text{out}}$. We can think of it as a cobordism $\tilde{\Sigma}_{\text{in}} \sqcup \overline{\tilde{\Sigma}_{\text{out}}} \to \tilde{\emptyset}_2$. Then we rename the labels of all the $-$ marked points in accordance with Lemma 2, and reverse the directions of the ribbons as needed, we also change the coupon to bring all the objects from the co-domain to domain by dualizing. Let this be $\tilde{M}^+ \colon \tilde{\Sigma}_{\text{in}}^+ \sqcup \overline{\tilde{\Sigma}_{\text{out}}}^+ \to \tilde{\emptyset}_2$

In Definition 4, we construct a map $\tilde{\mathcal{Z}}\left(\tilde{M}_+\right) : \mathcal{Z}\left(\tilde{\Sigma}_{\text{in}}^+ \sqcup \overline{\tilde{\Sigma}_{\text{out}}}^+\right) \to \tilde{\mathcal{Z}}\left(\tilde{\emptyset}\right)$, which by Lemma 1 and Lemma 2 is equivalent to a map $\tilde{\mathcal{Z}}\left(\tilde{\Sigma}_{\text{in}}\right) \otimes_{\mathbb{C}} \tilde{\mathcal{Z}}\left(\tilde{\Sigma}_{\text{out}}\right)^{\vee} \to \mathbb{C}$. We define $\tilde{\mathcal{Z}}\left(\tilde{M}\right)$ to be the induced map $\tilde{\mathcal{Z}}\left(\tilde{\Sigma}_{\text{in}}\right) \to \tilde{\mathcal{Z}}\left(\tilde{\Sigma}_{\text{out}}\right)$

Let $\Sigma^{g,k_+,k_-}$ be a surface with $g$ holes, with $k_+ + k_-$ boundary circles of which $k_+$ are incoming and $k_-$ are outgoing.

We using the facts that $Z\left(\Sigma^{0,k,1}\right)(A_1 \cdots \boxtimes A_k) = A_1 \otimes \cdots \otimes A_k$ and $Z\left(\Sigma^{0,1,0}\right)(A) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ below.

**Definition 4.** *Let $\tilde{M} \colon \tilde{\Sigma} \to \tilde{\emptyset}_2$ have a single coupon in it, suppose all points of $\tilde{\Sigma}$ are marked $+$ and all ribbons intersect the boundary. We define a morphism $\tilde{\mathcal{Z}}\left(\tilde{M}\right) : \tilde{\mathcal{Z}}\left(\tilde{\Sigma}\right) \to \mathbb{C}$. Let the marked points on $\tilde{\Sigma}$ be $A_1, \ldots, A_k$. Let $A_1, \ldots, A_{2n}$ be the labels which do not go into a coupon (even as each ribbon has 2 ends). So we assume for $i \leq n$, we have $A_{n+i}^{\vee} = A_i$. Define $M$ as in the previous section, we see $M$ is a cobordism from $\Sigma$ to $\Sigma_0$ where $\Sigma_0$ is the boundary created by cutting out tubes in $\tilde{M}$. So $\Sigma_0$ will consist of a $\Sigma^{0,2,0}$ for each non-coupon ribbon in $\tilde{M}$ and a single $\Sigma^{0,k-2n,0}$ coming from the coupon.*

*Thus $\mathcal{Z}(M) : \mathcal{Z}(\Sigma) \Rightarrow \mathcal{Z}(\Sigma_0)$. Also, we have $f \colon A_{k-2n+1} \otimes \cdots \otimes A_k \to \mathbb{1}$. Now, we want a map $\tilde{\mathcal{Z}}\left(\tilde{\Sigma}\right) \to \mathbb{C}$.*

$$\tilde{\mathcal{Z}}\left(\tilde{\Sigma}\right) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathcal{Z}(\Sigma)(A_1 \boxtimes \cdots \boxtimes A_k)) \cong \mathcal{Z}(\Sigma)(A_1 \boxtimes \cdots \boxtimes A_k) \xrightarrow{\mathcal{Z}(M)_{A_1 \boxtimes \cdots \boxtimes A_k}} \mathcal{Z}(\Sigma_0)(A_1 \boxtimes \cdots \boxtimes A_k)$$

*Now we use that fact that*

$$\begin{aligned} Z(\Sigma_0)(A_1 \boxtimes \cdots \boxtimes A_k) \cong & \mathcal{Z}\left(\Sigma^{0,k-2n,0}\right)(A_{k-2n+1} \boxtimes \cdots \boxtimes A_k) \otimes \bigotimes_{i=1}^{n} \mathcal{Z}\left(\Sigma^{0,2,0}\right)\left(A_i^{\vee} \boxtimes A_i\right) \\ = & \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A_{k-2n+1} \otimes \cdots \otimes A_k) \otimes \bigotimes_{i=1}^{n} \mathrm{Hom}_{\mathcal{C}}\left(\mathbb{1}, A_i^{\vee} \otimes A_i\right) \end{aligned}$$

*We now use the morphism*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A_{k-2n+1} \otimes \cdots \otimes A_k) \otimes \bigotimes_{i=1}^{n} \mathrm{Hom}_{\mathcal{C}}\left(\mathbb{1}, A_i^{\vee} \otimes A_i\right) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, f) \otimes \bigotimes_{i=1}^{n} \mathrm{Hom}_{\mathcal{C}}\left(\mathbb{1}, \mathrm{ev}_{A_i}\right)} \bigotimes_{i=1}^{n+1} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$$

We claim that $\tilde{\mathcal{Z}} \colon \mathrm{Cob}_{3,2}^{\mathrm{or}}(\mathcal{C}) \to \mathrm{Vect}_{\mathbb{C}}$ is a symmertic monoidal functor.

## References

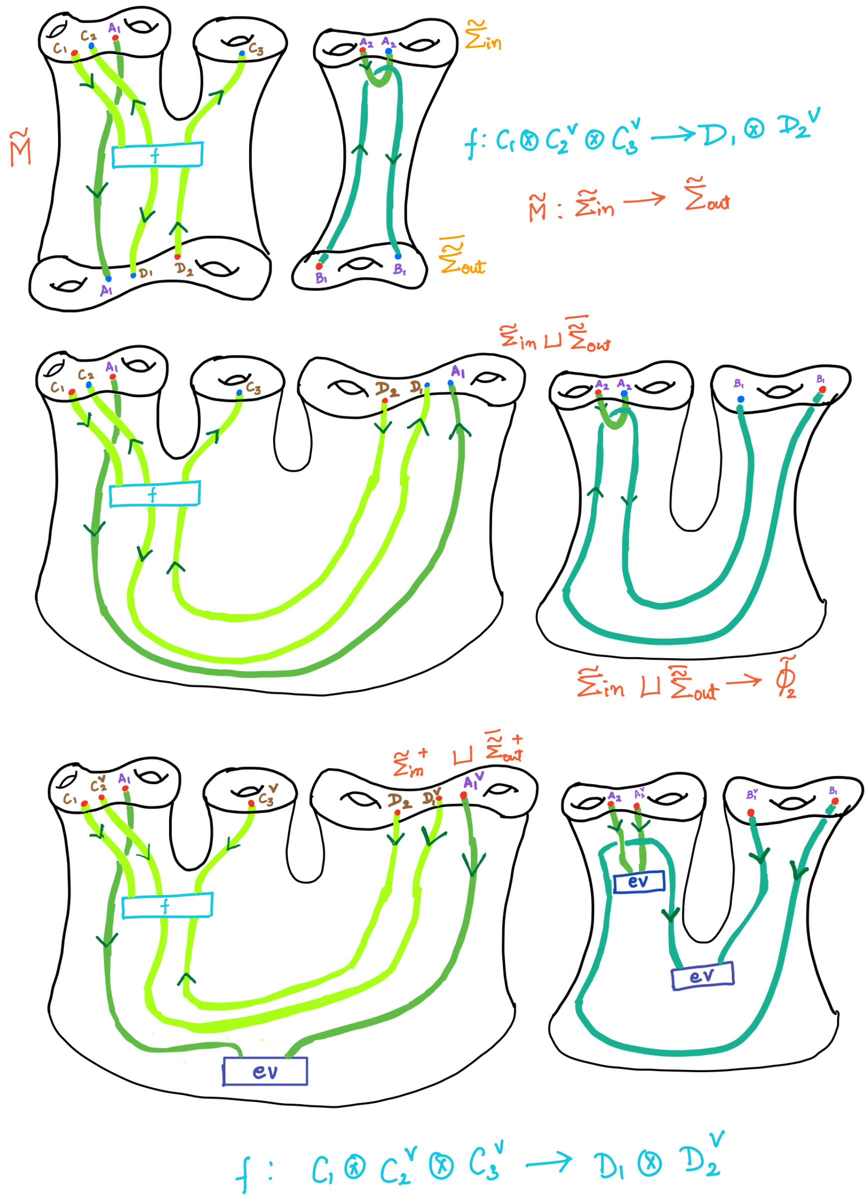


Figure 1: An illustrative example of starting with a cobordism in $\mathrm{Cob}^{\mathrm{or}}_{3,2}$ and using 2 reorienting the boundaries and relabelling the marked points.

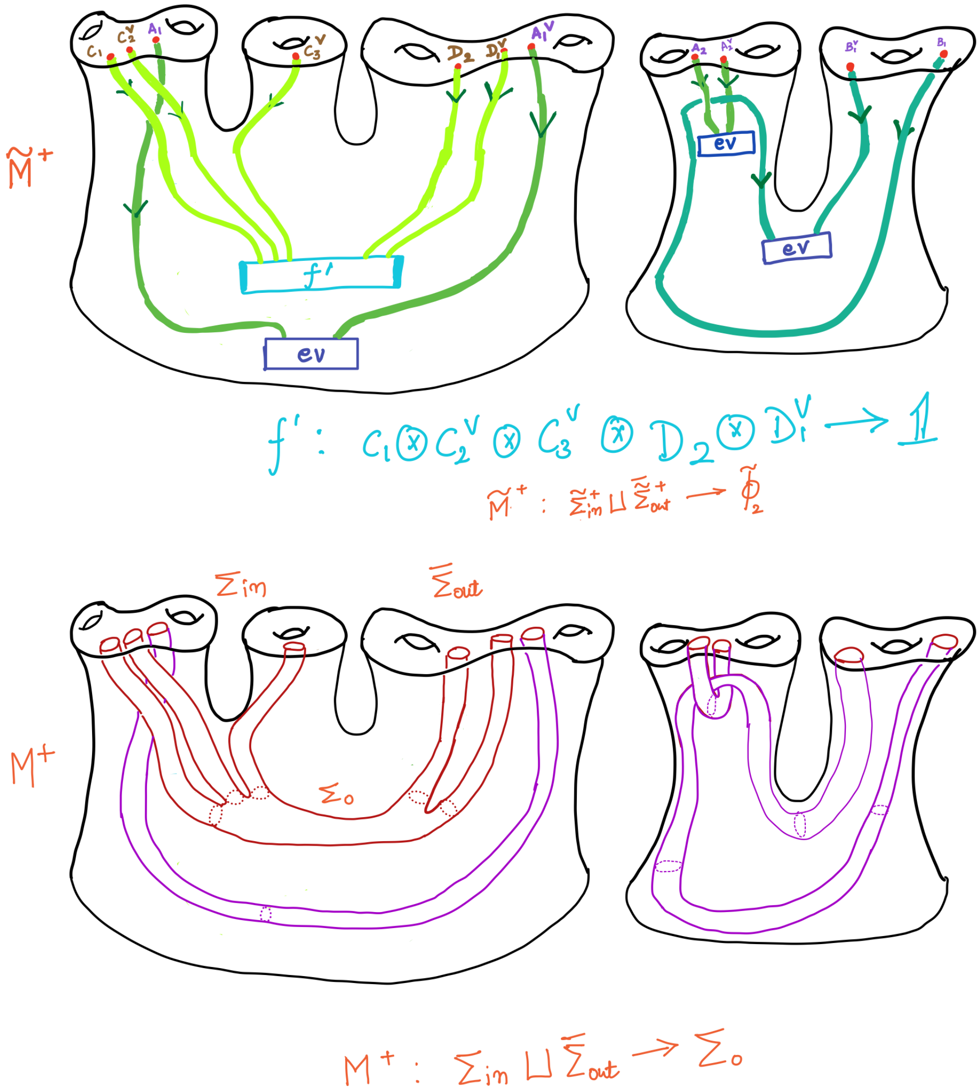


Figure 2: This step is turning the cobordism with a ribbon graph to cobordism in $\mathrm{Cob}^{\mathrm{or}}_{3,2,1}$ with discs and tubular neighbourhoods cut around marked points, ribbons and coupons.